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YANG-MILLS AMPLITUDES FROM STRING THEORY IN TWISTOR SPACE*

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Tree-level gluon scattering amplitudes in Yang-Mills theory frequently display simple mathematical structure which is completely obscure in the calculation of Feynman diagrams. We describe a novel way of calculating these amplitudes, motivated by a conjectured relation to twistor space, in which the problem of summing Feynman diagrams is replaced by the problem of solving a certain set of algebraic equations.

Keywords: Yang-Mills theory; twistors.

1. Yang-Mills Theory

As a successful model of the strong interactions, non-Abelian gauge theory (Yang-Mills theory) has formed the bedrock of the Standard model of particle physics for over thirty years. Although difficult to analyze at low energies, Yang-Mills theory is asymptotically free, allowing reliable perturbative calculations at the energies accessible in today's particle accelerators, far above the QCD scale $\Lambda \approx 200$ MeV.

Since covariant rules for the Feynman diagrammatic calculation of gluon scattering amplitudes in Yang-Mills theory have been known for decades and can now be found in any modern textbook on quantum field theory, one might have thought that any perturbative quantity of experimental interest had long ago been calculated to the desired accuracy, but this is far from the case[4]. Already at tree level, calculations are complicated by the fact that the number of Feynman diagrams contributing to an n-particle amplitude is very large, growing faster than factorially with n. Moreover, since the non-Abelian gauge boson has a complicated set of cubic and quartic self-interactions, each individual diagram can lead to very lengthy functions of the kinematic variables. The story becomes even more complicated beyond tree level, but that it outside the scope of this note.

Much of the complication stems from the fact that although gauge invariance imposes tight constraints on the full amplitudes, these constraints are completely obscured in the Feynman diagrammatic calculation. Gauge invariance does not manifest itself in individual diagrams but rather leads to substantial cancellations when

*Based on work done in collaboration with Radu Roiban and Anastasia Volovich.

the many diagrams are summed. This problem is exacerbated when amplitudes are expressed in terms of inefficient kinematic variables. The momentum and polarization four-vectors p_i^μ and ϵ_i^μ (where $i = 1, \ldots, n$ label the n external gluons) are inefficient variables because they possess a large amount of redundancy: $p_i^2 = 0$ for on-shell gluons, and gauge invariance implies that $p_i \cdot e_i = 0$ and that e_i is physically equivalent to $e_i + \alpha p_i$ for any real number α. A much more efficient choice of variables employs the spinor helicity notation, in which we decompose each null momentum p_i^μ into a pair of commuting two-component spinors via $p_i^\mu = (\sigma^\mu)_{\alpha\dot\alpha}\lambda^\alpha\widetilde\lambda^{\dot\alpha}$, where σ^μ are the Pauli matrices. Furthermore, the polarization ϵ_i^μ is determined uniquely (up to a gauge transformation) by making a choice $(\pm)_i$ for the helicity of the corresponding gluon.

Explicit calculation reveals that in many cases the final result for the tree-level n-gluon amplitude A_n has a simple form which is belied by the complexity of intermediate steps in the calculation. The most striking example of this simplification occurs for so-called MHV (maximally helicity violating) amplitudes, for which two gluons have negative helicity and the remaining $n-2$ gluons have positive helicity. When the Feynman diagrams are summed up and the dust settles, one finds the remarkable Parke-Taylor formula[1,2]

$$A_n(\lambda_i^\alpha, \widetilde\lambda_i^{\dot\alpha}) = \langle\lambda_r, \lambda_s\rangle^4 \prod_{i=1}^n \frac{1}{\langle\lambda_i, \lambda_{i+1}\rangle}, \tag{1}$$

where we suppress the overall delta-function $\delta^4(\sum_{i=1}^n \lambda_i^\alpha\widetilde\lambda_i^{\dot\alpha})$ of momentum conservation, g is the Yang-Mills coupling constant, $r, s \in \{1, \ldots, n\}$ label the two negative-helicity gluons, the subscript i is understood cyclically (i.e. $n+1 \equiv 1$), and we use the standard bracket inner product $\langle\lambda_i, \lambda_j\rangle = \epsilon_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta$. Analogous formulas become progressively more complicated as more negative helicity gluons are added.

One the one hand, the unexpected simplification of the amplitudes is a wonderful thing: it means that we can pass simple formulas rather than complicated ones to our experimentalist friends. On the other hand, as theorists the existence of simple formulas such as (1) is a smoking gun which should alert us to the fact that the traditional Feynman diagrammatic calculation of perturbative scattering amplitudes seems to be completely obscuring some rich structure in the theory. Clearly, it would be useful to unlock this structure.

2. Amplitudes from Twistor Space

An important step in this direction was taken by E. Witten[5], who proposed that this structure could be explained by an equivalence between Yang-Mills theory and a certain topological string theory in twistor space. Witten's proposal provided a context for an earlier observation by V. P. Nair that the MHV amplitude (1) could be understood in terms of an underlying current algebra. Witten's twistor string theory was used by R. Roiban, A. Volovich and the author[6,7,8] to derive a novel formula for tree-level n-gluon scattering amplitudes which are remarkable for their lack of any apparent relation to traditional Feynman diagrams.

Starting from given $(\lambda_i^\alpha, \widetilde{\lambda}_i^{\dot\alpha}, (\pm)_i)$ describing the n gluons, one looks for all solutions (which in general are complex) to the $2n+2d+2$ polynomial equations

$$0 = \sum_{i=1}^{n} \xi_i (\sigma_i)^k \widetilde{\lambda}_i^{\dot\alpha}, \qquad 0 = \lambda_i^\alpha - \xi_i \sum_{k=0}^{d} (\sigma_i)^k a_k^\alpha, \tag{2}$$

in the $2n+2d+2$ variables σ_i, ξ_i and a_k^α $(k = 0, \ldots, d)$, where d is equal to the number of negative helicity gluons minus one. It is easy to see that there can be no solutions unless $0 = \sum_{i=1}^n \lambda_i^\alpha \widetilde{\lambda}_i^{\dot\alpha}$, which we recognize as overall momentum conservation. Moreover, if there is one solution then there is an infinite family related by certain $GL(2)$ transformations. Modulo this ambiguity, which we can fix (for example) by setting any three of the σ's and any one of the a's to any desired value, the equations (2) always have a finite number of roots.

If we assemble the right-hand sides of the equations (2) into a $2n+2d+2$ component vector H_r and the variables $(\sigma_i, \xi_i, a_k^\alpha)$ into one $2n+2d+2$ component vector q_s, then the prescription for the tree-level n-gluon amplitude is simply

$$A_n(\lambda_i^\alpha, \widetilde{\lambda}_i^{\dot\alpha}, (\pm)_i) = i(2\pi)^4 g^{n-2} \sum_{q_s : H_r(q_s)=0} \mathcal{J} \left(\prod_{i=1}^{n} \frac{1}{\xi_i(\sigma_i - \sigma_{i+1})} \right) \frac{(\det F_{i'k})^4}{\det(\partial H_r / \partial q_s)}, \tag{3}$$

where $F_{i'k} = \xi_{i'}(\sigma_{i'})^k$ is a $(d+1)\times(d+1)$ matrix, with $k = 0, \ldots, d$ and i' running over the subset of $\{1, \ldots, n\}$ corresponding only to the $d+1$ negative helicity gluons. Finally the quantity $\mathcal{J}$ is a Jacobian which arises from fixing the $GL(2)$ symmetry and can be calculated using the familiar Fadeev-Popov procedure. For example, if we choose to fix σ_i, σ_j, σ_k and a_k^α then $\mathcal{J} = a_k^\alpha(\sigma_i - \sigma_j)(\sigma_j - \sigma_k)(\sigma_k - \sigma_i)$.

The formula (3) agrees with previously known results in all cases which can be checked, but it has not been explicitly proven. Moreover, it passes a number of non-trivial checks[8] which indicate that (3) indeed encapsulates the complete tree-level n-particle S-matrix of Yang-Mills theory. We note in conclusion that an alternate set of equations can be formulated which have fewer variables but higher degree[8].

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