International Journal of Modern Physics A

Semileptonic decays of D, B, K mesons and the full CKM matrix from unquenched lattice QCD

Masataka Okamoto
(for the Fermilab Lattice, MILC, and HPQCD Collaborations)
Fermi National Accelerator Laboratory
Batavia, Illinois 60510, USA

Received (Day Month Year)
Revised (Day Month Year)

We use lattice QCD to fully determine the CKM matrix. $|V_{cd}|$, $|V_{cs}|$, $|V_{ub}|$, $|V_{cb}|$ and $|V_{us}|$ are, respectively, directly determined with our lattice results for form factors of semileptonic $D \to \pi l\nu$, $D \to K l\nu$, $B \to \pi l\nu$, $B \to D l\nu$ and $K \to \pi l\nu$ decays. The accuracy is comparable to that of the Particle Data Group averages. In addition, $|V_{ud}|$, $|V_{tb}|$, $|V_{ts}|$ and $|V_{td}|$ are determined by using unitarity of the CKM matrix and the experimental result for $\sin(2\beta)$. In this way, we obtain all 9 CKM matrix elements, where the only theoretical input is lattice QCD.

Keywords: semileptonic decay; CKM matrix; lattice QCD.

1. Introduction

The Cabibbo-Kobayashi-Maskawa (CKM) matrix, which relates the mass eigenstates and the weak eigenstates in the Standard Model electroweak theory, is a set of parameters. To determine each CKM matrix element, one requires both theoretical and experimental inputs. On the theoretical side, one needs to know relevant hadronic amplitudes, which often contain nonperturbative QCD effects. A major role of lattice QCD is to calculate such hadronic amplitudes reliably and accurately, from first principles. One can then extract the CKM matrix elements by combining lattice QCD as the theoretical input with the experimental input such as decay rates. In this paper, we show that it is now possible to *fully* determine the CKM matrix, for the first time, using lattice QCD. The result for the full CKM matrix with lattice QCD is:

$$V_{\mathrm{CKM}} = \begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ 0.9744(5)(3) & 0.225(2)(1) & 3.5(5)(5)\times 10^{-3} \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ 0.24(3)(2) & 0.97(10)(2) & 3.9(1)(3)\times 10^{-2} \\ |V_{td}| & |V_{ts}| & |V_{tb}| \\ 8.1(2.7)\times 10^{-3} & 3.8(4)(3)\times 10^{-2} & 0.9992(0)(1) \end{pmatrix} \quad (1)$$

where the first errors are from lattice calculations and the second are experimental, except the one for $|V_{td}|$ which is a combined lattice and experimental error. The results for the Wolfenstein parameters with lattice QCD are:

$$\lambda = 0.225(2)(1), \quad A = 0.77(2)(7), \quad \rho = 0.16(28), \quad \eta = 0.36(11). \tag{2}$$

To directly determine 5 CKM matrix elements ($|V_{cd}|$, $|V_{cs}|$, $|V_{ub}|$, $|V_{cb}|$ and $|V_{us}|$), we use 5 semileptonic decays ($D \to \pi l\nu$, $D \to K l\nu$, $B \to \pi l\nu$, $B \to D l\nu$ and $K \to \pi l\nu$), for which the techniques for lattice calculations are well established, and thus reliable calculations are possible. The accuracy of previous lattice calculations was limited by two large systematic uncertainties — the error from the "quenched" approximation (neglect of virtual quark loop effects) and the error from the "chiral" extrapolation in light quark mass ($m_l \to m_{ud}$). Both led to effects of around 10–20%. Our present work [1,2] successfully reduces these two dominant uncertainties. The error from the quenched approximation is removed by using the MILC unquenched gauge configurations [3], where the effect of u, d and s quarks is included ($n_f = 2+1$). The error from the chiral extrapolation is greatly reduced by using improved staggered quarks. With this improved approach, the accuracy of the 5 CKM matrix elements is comparable to that of the Particle Data Group [4]. The results for $|V_{ub}|$, $|V_{cb}|$ and $|V_{us}|$ are preliminary. We then use CKM unitarity to determine the other 4 CKM matrix elements ($|V_{ud}|$, $|V_{tb}|$, $|V_{ts}|$ and $|V_{td}|$). In this way, we obtain all 9 CKM matrix elements and all the Wolfenstein parameters, where the only theoretical input is lattice QCD (assuming that the Standard Model is correct).

This paper is organized as follows. In Sec. 2 we present our results for the 5 semileptonic decays and the 5 CKM matrix elements. In Sec. 3 we first check unitarity of the charm row of the CKM matrix, and then give the results for the other 4 CKM matrix elements from unitarity, as well as the Wolfenstein parameters. In Sec. 4 we give a summary. The results for the D and B decays have been presented in Refs. [1,2]. This work is a part of our ongoing project of heavy quark physics with lattice QCD; see also [5,6].

2. 5 CKM matrix elements from 5 semileptonic decays

2.1. $D \to \pi(K) l\nu$, $|V_{cd(s)}|$ *and* $B \to \pi l\nu$, $|V_{ub}|$

The differential decay rate $d\Gamma/dq^2$ for the heavy-to-light semileptonic decay $H \to P l\nu$ is proportional to $|V_{ij}|^2 |f_+(q^2)|^2$, where f_+ is a form factor of the relevant hadronic amplitude defined through

$$\begin{aligned} \langle P|V^\mu|H\rangle &= f_+(q^2)(p_H + p_P - \Delta)^\mu + f_0(q^2)\Delta^\mu \\ &= \sqrt{2m_H}\,\left[v^\mu\, f_\parallel(E) + p_\perp^\mu\, f_\perp(E)\right]. \end{aligned} \tag{3}$$

Here $q = p_H - p_P$, $\Delta^\mu = (m_H^2 - m_P^2)\, q^\mu/q^2$, $v = p_H/m_H$, $p_\perp = p_P - Ev$ and $E = E_P$. To determine the CKM matrix element $|V_{ij}|$ with the experimental rate $\int_{q^2_{\rm min}}^{q^2_{\rm max}} dq^2\ (d\Gamma/dq^2)$, we calculate $f_{+,0}$ as a function of q^2. Below we briefly describe our analysis procedure [1,2].

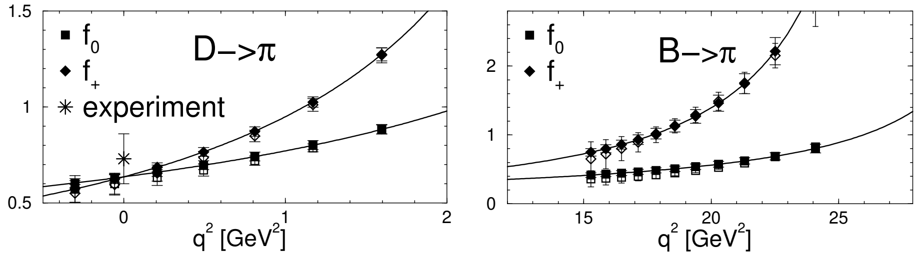

Fig. 1. Form factors from BK-based (filled symbols and curves) and non-BK-based (open symbols) analyses for $D \to \pi l \nu$ (left panel) and $B \to \pi l \nu$ (right) decays.

We first extract the form factors $f_\parallel$ and $f_\perp$, and interpolate and extrapolate the results for $f_\parallel$ and $f_\perp$ to common values of E using the parametrization of Becirevic and Kaidalov (BK) [7]. We then perform the chiral extrapolation ($m_l \to m_{ud}$) at each E using the NLO correction in staggered chiral perturbation theory [8]. Finally we convert the results for $f_\perp$ and $f_\parallel$ at $m_l = m_{ud}$, to f_+ and f_0. To extend f_+ and f_0 to functions of q^2, we again make a fit using the BK parameterization [7],

$$f_+(q^2) = \frac{f_+}{(1-\tilde{q}^2)(1-\alpha\tilde{q}^2)}, \quad f_0(q^2) = \frac{f_+}{1-\tilde{q}^2/\beta}, \tag{4}$$

where $\tilde{q}^2 = q^2/m_{H^*}^2$. We obtain [1,2]

$$f_+^{D\to\pi} = 0.64(3), \quad \alpha^{D\to\pi} = 0.44(4), \quad \beta^{D\to\pi} = 1.41(6), \tag{5}$$

$$f_+^{D\to K} = 0.73(3), \quad \alpha^{D\to K} = 0.50(4), \quad \beta^{D\to K} = 1.31(7), \tag{6}$$

for the D decay, and

$$f_+^{B\to\pi} = 0.23(2), \quad \alpha^{B\to\pi} = 0.63(5), \quad \beta^{B\to\pi} = 1.18(5), \tag{7}$$

for the B decays, where the errors are statistical only. To estimate the error from the BK parameterization, *i.e.,* the error for q^2 dependence, we also make an alternative analysis, where we perform a 2-dimensional polynomial fit in $\big(m_l, E(q^2)\big)$. A comparison between two analyses is shown in Fig. 1. The results for D decays agree well with recent experimental results [9].

We now determine the CKM matrix elements by integrating $|f_+(q^2)|^2$ over q^2 and using experimental decay rates [4,10,11]. For $|V_{ub}|$ we use a combined average of the decay rate for $q^2 \geq 16$ GeV^2 in Refs. [10] and [11]. We obtain

$$|V_{cd}| = 0.239(10)(24)(20) \quad , \quad |V_{cs}| = 0.969(39)(94)(24) \tag{8}$$

from the D decay, and

$$|V_{ub}| \times 10^3 = 3.48(29)(38)(47) \tag{9}$$

from the B decays, where the first errors are statistical, the second systematic, and the third are experimental errors from the decay rates. The systematic errors are dominated by the finite lattice spacing effects, *i.e.,* the lattice discretization effects; see Table 1. The results for the CKM matrix elements agree with the Particle Data Group averages [4] with a comparable accuracy.

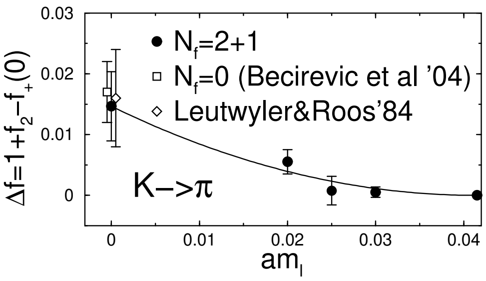

Fig. 2. m_l-dependence of Δf for $K \to \pi l \nu$ decay, together with results from Refs. [14,15].

2.2. $B \to Dl\nu$, $|V_{cb}|$ and $K \to \pi l \nu$, $|V_{us}|$

The form factors of $B \to Dl\nu$ and $K \to \pi l\nu$ decays can be calculated more accurately than those of heavy-to-light ($H \to Pl\nu$) decays, due to the symmetry between the initial and final states. The differential decay rate of $B \to Dl\nu$ is proportional to the square of $|V_{cb}|\mathcal{F}(w)$, where $\mathcal{F}(w)$ is a linear combination of $h_+(w)$ and $h_-(w)$ defined through

$$\langle D|V^\mu|B\rangle \;=\; \sqrt{m_B m_D} \times [h_+(w)(v+v')^\mu + h_-(w)(v-v')^\mu], \tag{10}$$

with $v = p_B/m_B$, $v' = p_D/m_D$ and $w = v \cdot v'$. To extract $|V_{cb}|$, we calculate the form factor at $w = 1$, $\mathcal{F}(1)$, by employing the double ratio method [12]. The light quark mass dependence of $\mathcal{F}(1)$ is mild, and by extrapolating the result linearly to $m_l \to 0$ we obtain [2]

$$\mathcal{F}^{B\to D}(1) = 1.074(18)(16), \tag{11}$$

where the first error is statistical, and the second is systematic, as summarized in Table 1. Using Eq. (11) and the experimental result for $|V_{cb}|\mathcal{F}(1)$ [13], we obtain

$$|V_{cb}| \times 10^2 = 3.91(07)(06)(34), \tag{12}$$

where the errors from the lattice calculation (first two) are much smaller than the experimental one (third).

Finally we study the $K \to \pi l\nu$ decay to determine $|V_{us}|$. The expression for the $K \to \pi$ decay amplitude is given in an analogous way to Eq. (3). We calculate

Table 1. Systematic errors in lattice calculations. For comparison, the error for each CKM matrix element by the Particle Data Group [4] is shown in the last row.

semileptonic decay CKM matrix element	$D \to \pi(K)l\nu$ $\lvert V_{cd(s)}\rvert$	$B \to \pi l\nu$ $\lvert V_{ub}\rvert$	$B \to Dl\nu$ $\lvert V_{cb}\rvert$	$K \to \pi l\nu$ $\lvert V_{us}\rvert$
q^2 dependence	2%	4%		<1%
$m_l \to m_{ud}$ extrapolation	3%(2%)	4%	1%	1%
operator matching	<1%	1%	1%	<1%
discretization effects	9%	9%	<1%	
total systematic error	10%	11%	2%	1%
error in PDG average	5%(1%)	13%	4%	1%

the $K^0 \to \pi^-$ form factor at $q^2 = 0$, $f_+(0) = f_0(0)$, by employing the three steps method, as in Ref. [14]. To perform the $m_l \to m_{ud}$ extrapolation for $f_+(0)$, we subtract the leading logarithmic correction f_2 in chiral perturbation theory, *i.e.,* define $\Delta f \equiv 1 + f_2 - f_+(0)$. We make a fit to Δf adopting an ansatz, $\Delta f = (A + Bm_l)(m_s - m_l)^2$, where A, B are fit parameters. As an exploratory study of $f_+^{K\to\pi}$, we use an improved staggered action for u, s quarks and an improved Wilson action for the d quark. The m_l-dependence of Δf and the extrapolated result are shown in Fig. 2, together with a recent quenched lattice result [14] and an earlier result by Leutwyler and Roos [15]. Our preliminary result is $\Delta f = 0.015(6)(9)$, giving

$$f_+^{K^0\to\pi^-}(0) = 0.962(6)(9), \tag{13}$$

which agrees well with those of Refs. [14,15]. Combining with a recent experimental result for $|V_{us}|f_+(0)$ [16], we obtain

$$|V_{us}| = 0.2250(14)(20)(12). \tag{14}$$

3. Other 4 CKM matrix elements using unitarity and Wolfenstein parameters

Having the 5 CKM matrix elements directly determined from the 5 semileptonic decays, we can check unitarity of the second row of the CKM matrix. Using Eqs. (8) and (12), we get

$$(|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2)^{1/2} = 1.00(10)(2), \tag{15}$$

which is consistent with unitarity. Hereafter the first error is from the lattice calculation and the second is experimental, unless otherwise stated.

We now use unitarity of the CKM matrix to determine the other 4 CKM matrix elements. $|V_{ud}|$, $|V_{tb}|$ and $|V_{ts}|$ are easily determined:

$$|V_{ud}| = (1 - |V_{us}|^2 - |V_{ub}|^2)^{1/2} = 0.9744(5)(3), \tag{16}$$

$$|V_{tb}| = (1 - |V_{ub}|^2 - |V_{cb}|^2)^{1/2} = 0.9992(0)(1), \tag{17}$$

$$|V_{ts}| = |V_{us}^* V_{ub} + V_{cs}^* V_{cb}| \;/\; |V_{tb}| \simeq |V_{cs}^* V_{cb}| \;/\; |V_{tb}| = 3.8(4)(3) \times 10^{-2}. \tag{18}$$

Before proceeding to extract $|V_{td}|$, let us give some of the Wolfenstein parameters. Using Eqs. (14), (12) and (9), we get

$$\lambda = |V_{us}| = 0.225(2)(1), \tag{19}$$

$$A = |V_{cb}|/\lambda^2 = 0.77(2)(7), \tag{20}$$

$$(\rho^2 + \eta^2)^{1/2} = |V_{ub}|/(A\lambda^3) = 0.40(6)(6). \tag{21}$$

It is difficult to extract $|V_{td}|$ accurately using unitarity alone, because V_{td} $(= A\lambda^3(1-\rho-i\eta)$ in Wolfenstein parameterization) contains a large CP-violating phase and its magnitude is small. We use here the experimental result for $\sin(2\beta)$ from

$B \to (c\bar{c})K^{(*)}$ decays. By performing a unitary triangle analysis with $\sin(2\beta) = 0.726(37)$ [13] and Eq. (21), we obtain

$$\rho \;=\; 0.16(28), \qquad \eta \;=\; 0.36(11), \tag{22}$$

$$|V_{td}| = 8.1(2.7)\times 10^{-3} \tag{23}$$

with a combined lattice and experimental error, completing the full CKM matrix.

4. Summary

In this paper, we have shown that it is indeed possible to fully determine the CKM matrix using lattice QCD. $|V_{cd}|$, $|V_{cs}|$, $|V_{ub}|$, $|V_{cb}|$ and $|V_{us}|$ are directly determined with our lattice results for form factors of 5 semileptonic decays. The error from the quenched approximation is removed by using the MILC unquenched gauge configurations. The error from the $m_l \to m_{ud}$ extrapolation is significantly reduced by using improved staggered quarks. The accuracy is comparable to that of the Particle Data Group averages, as seen in Table 1. The other 4 CKM matrix elements $|V_{ud}|$, $|V_{tb}|$, $|V_{ts}|$ and $|V_{td}|$ are then determined by using unitarity of the CKM matrix and the experimental result for $\sin(2\beta)$. In this way, we obtain the full CKM matrix, Eq. (1), and the Wolfenstein parameters, Eq. (2). We emphasize that the only theoretical input in this paper is lattice QCD.

Acknowledgments

We thank the authors in Ref. [1]. We thank Andreas S. Kronfeld, Jack Laiho, and Robert Sugar for valuable comments on the manuscript. This work is supported by the Fermilab Computing Division, the SciDAC program, and URA.

References

1. C. Aubin *et al.*, arXiv:hep-ph/0408306, to appear in Phys. Rev. Lett.
2. M. Okamoto *et al.*, arXiv:hep-lat/0409116, and work in progress.
3. C. Bernard *et al.*, Phys. Rev. D **64**, 054506 (2001); C. Aubin *et al.*, **70**, 094505 (2004).
4. S. Eidelman *et al.* [Particle Data Group Collaboration], Phys. Lett. B **592**, 1 (2004).
5. J. N. Simone *et al.*, arXiv:hep-lat/0410030.
6. M. di Pierro *et al.*, Nucl. Phys. Proc. Suppl. **129**, 328 (2004); Nucl. Phys. Proc. Suppl. **129**, 340 (2004); M. Okamoto *et al.*, Nucl. Phys. Proc. Suppl. **129**, 334 (2004).
7. D. Becirevic and A. B. Kaidalov, Phys. Lett. B **478**, 417 (2000).
8. C. Aubin and C. Bernard, arXiv:hep-lat/0409027, and work in progress.
9. M. Ablikim *et al.* [BES Collab.], hep-ex/0406028; G. S. Huang *et al.* [CLEO Collab.], hep-ex/0407035; J. M. Link *et al.* [FOCUS Collab.], hep-ex/0410037.
10. S. B. Athar *et al.* [CLEO Collaboration], Phys. Rev. D **68**, 072003 (2003).
11. K. Abe *et al.* [Belle Collaboration], hep-ex/0408145.
12. S. Hashimoto *et al.*, Phys. Rev. D **61**, 014502 (2000).
13. Heavy Flavor Averaging Group, `http://www.slac.stanford.edu/xorg/hfag/`.
14. D. Becirevic *et al.*, arXiv:hep-ph/0403217.
15. H. Leutwyler and M. Roos, Z. Phys. C **25**, 91 (1984).
16. T. Alexopoulos *et al.* [KTeV Collaboration], Phys. Rev. Lett. **93**, 181802 (2004).