One Ring to Rule Them All

Based upon: **Iosif Bena and NPW** **hep-th/0408106**

Related recent papers:

- Elvang, Emparan, Mateos and Reall: hep-th/0408120
- Gauntlett and Gutowski: hep-th/0408122

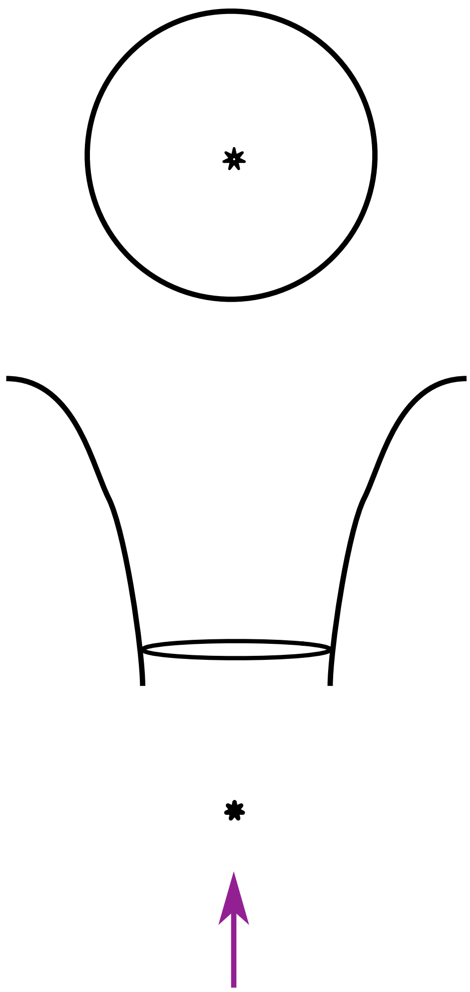
Outline

- **Motivation: Why study black rings in five dimensions?**
  - **Mathur's conjecture**
  - **Black hole uniqueness**
  - **Classification of (non-compact) supersymmetric backgrounds with fluxes**
- **The general 3-charge solution**
  - **M-theory version of the problem**
  - **Solving using the supersymmetries**
  - **Complete reduction of the problem to a *linear* system in four-dimensional electromagnetism**
- **Example**
  - **The general round ring**
- **Conclusions**

Mathur's Conjecture

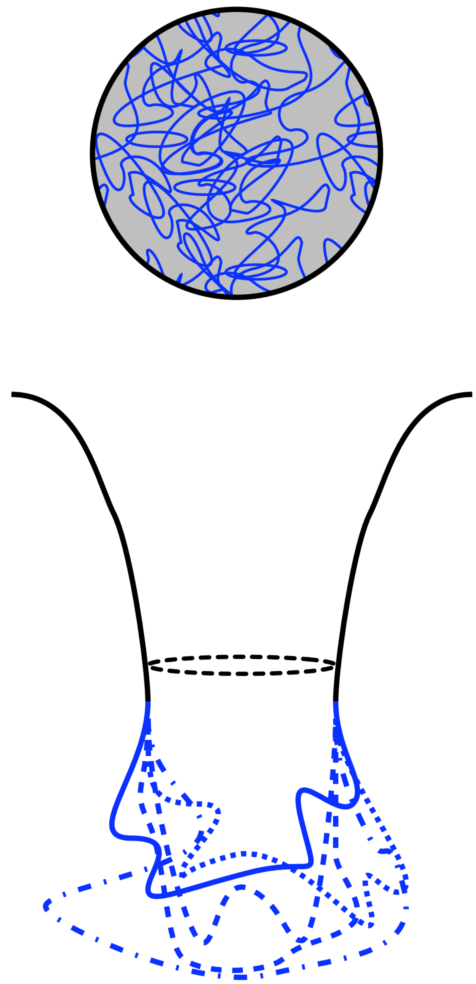
Lunin and Mathur: hep-th/0105126, hep-th/0109154, hep-th/0202072, hep-th/0211292, hep-th/0401115

Conventional, classical picture:



Strong quantum effects, no semi-classical description
BUT No Information is lost!

Mathur's picture:



Many, many ***smooth*** supertube microstates ⇒

Information is semi-classical

Black hole hair ⟷ **The set of smooth supertube solutions**

Horizon ⟷ **Effect coarse graining over supertube microstates**

Evidence: Based mainly upon extremal, and near extremal, D1-D5 system: Supertube solutions, scattering, F1-P system, intriguing physical picture ......

Testing Mathur's Conjecture

Issues:

- D1-D5 system is algebraically very special, essentially harmonic.

  T-dual to F1-P system

- Extremal D1-D5 system has no macroscopic event horizon

- Black hole entropy counting with D-branes is based upon three-charge black hole (in five-dimensions): the D1-D5-P system

Strominger and Vafa, hep-th/9601029

$$Horizon\ \ Area \sim \sqrt{Q_1 Q_2 Q_3}$$

For it to be compelling, Mathur's conjecture must work for the three charge black hole

$\Longrightarrow$ **We need to classify all three-charge black rings and supertubes in five dimensions.**

Then:

- Which classical supertubes are single microstates, and not superpositions of microstates.
- Are there sufficiently many smooth three-charge solutions to account for the microstates?
- How do you count them? What are the distinct microstates?
  Practical Issue: linearity of underlying equations is important.
- Black hole/ring uniqueness in five dimensions? (**No!**)
  How badly violated?

Some Recent References:

Mathur, Saxena and Srivastava: hep-th/0311092

Mathur: hep-th/0401115

Bena and Kraus: hep-th/0402144

Palmer and Marolf: hep-th/0403025

Lunin: hep-th/0404006

Bena: hep-th/0404073

Bak, Hyakutake and Ohta: hep-th/0404104

Giusto, Mathur and Saxena: hep-th/0405017, hep-th/0406103

Elvang, Emparan, Mateos and Reall: hep-th/0407065, hep-th/0408120

Bak, Hyakutake, Kim and Ohta: hep-th/0407253

Bena and Warner: hep-th/0408106

Gauntlett and Gutowski: hep-th/0408010, hep-th/0408122

Bena and Krauss: hep-th/0408186

Three-Charge, BPS Black Rings and Supertubes in M-theory

Three asymptotic electric charges: **Q_1, Q_2, Q_3** M2-branes

	1	2	3	4	5	6	7	8	9	10	11
Q_1	*	*	*								
Q_2	*			*	*						
Q_3	*					*	*				

time (1); T^2 (2–3); T^2 (4–5); T^2 (6–7); R^4 (8–11)

compactification directions (T^2, T^2, T^2) **space transverse to branes** (R^4)

Ring: Three magnetic dipole charges, **q_1, q_2, q_3**, M5-branes

	1	2	3	4	5	6	7	8	9	10	11
q_1	*			*	*	*	*				
q_2	*	*	*	*	*						
q_3	*	*	*			*	*				

Directions 8–11: $y^a = \mu^a(\phi)$

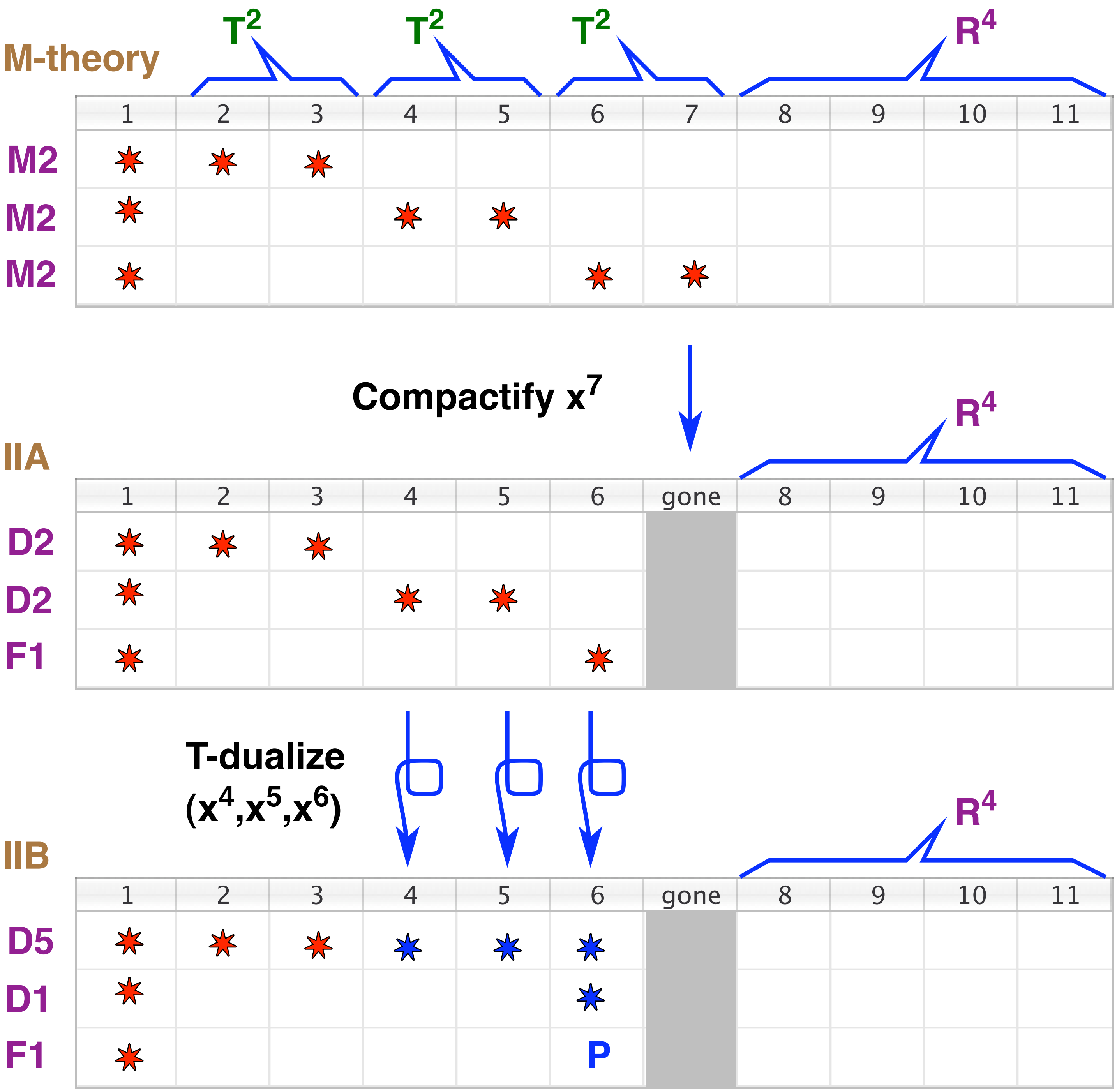
Fifth spatial direction of M5-branes defined by loops in **R^4**:

$$y^a = \mu^a_{(j)}(\phi) \,, \quad 0 \le \phi < 2\pi \,, \quad j = 1, 2, 3$$

Loops of magnetically charged, black strings

- Closed loop $\Rightarrow$ Dipolar fields
- **Completely arbitrary "Ring Shape"** $\Rightarrow$ Four arbitrary functions

Relation to D1-D5-P system in IIB



The Problem in M-theory

Equations of motion:

$$R_{\mu\nu} + R\, g_{\mu\nu} = \tfrac{1}{12} F_{\mu\rho\lambda\sigma} F_{\nu}{}^{\rho\lambda\sigma}$$

$$d * F + \tfrac{1}{2} F \wedge F = 0$$

Supersymmetry conditions:

$$\delta\psi_{\mu} \equiv \nabla_{\mu}\varepsilon + \tfrac{1}{288} F_{\nu\rho\lambda\sigma}\left(\Gamma_{\mu}{}^{\nu\rho\lambda\sigma} - 8\,\delta_{\mu}^{\nu}\Gamma^{\rho\lambda\sigma}\right)\varepsilon = 0$$

Naive difficulty is that in three-charge/three-dipole-charge problem, the $\mathbf{F \wedge F}$ term is non-zero

$\Rightarrow$ The problem is intrinsically non-linear

D1-D5 is intrinsically simpler: Harmonic Ansatz works

However, the three-charge/three-dipole-charge problem is linear when organized in the right way ....

Best approach: Determine the supersymmetries first ... then solve the supersymmetry variations for the metric and the Maxwell fields.

C. Gowdigere, D. Nemeschansky, K. Pilch and NPW
hep-th/0306097; hep-th/0306098; hep-th/0403005; hep-th/0403006

The Supersymmetries

	1	2	3	4	5	6	7	8	9	10	11
M2	*	*	*								
M2	*			*	*						
M2	*					*	*				

We want ring solutions that are *mutually BPS* with the the BMPV black hole $\Rightarrow$ the space of supersymmetries is defined by:

$$\frac{1}{2}\left(\mathbb{1} + \Gamma^{123}\right)\varepsilon = \frac{1}{2}\left(\mathbb{1} + \Gamma^{145}\right)\varepsilon = \frac{1}{2}\left(\mathbb{1} + \Gamma^{167}\right)\varepsilon = 0$$

- Four supersymmetries
- On the transverse $\mathbb{R}^4$: $\frac{1}{2}\left(\mathbb{1} - \Gamma^{89\,10\,11}\right)\varepsilon = 0$
- The vector: $K^\mu \equiv \bar{\varepsilon}\Gamma^\mu\varepsilon$
  - is a timelike Killing vector
  - $K^\mu = (1,0,0,\ldots,0)$

$\Rightarrow$ The Killing spinor is normalized by $\varepsilon = e^{\frac{1}{2}B_0}\,\varepsilon_0$

where ϵ_0 is a constant spinor, and the metric has the form:

$$ds^2 = -e^{2B_0}dt^2 + \ldots$$

$\Rightarrow$ **The supersymmetries are fixed by the physics**

Constraining the Ansatz using brane probes

$$e^1 = e^{B_0}\left(dx^1 + \vec{\omega}\cdot d\vec{y}\right),$$

T^2: $e^2 = e^{B_1}dx^2, \qquad e^3 = e^{B_1}dx^3,$

T^2: $e^4 = e^{B_2}dx^4, \qquad e^5 = e^{B_2}dx^5,$

T^2: $e^6 = e^{B_3}dx^6, \qquad e^7 = e^{B_3}dx^7, \qquad e^{7+a} = e^{B_4}dy^a$ (R^4)

$$C^{(3)} = -e^1\wedge e^2\wedge e^3 - e^1\wedge e^4\wedge e^5 - e^1\wedge e^6\wedge e^7 + \\ + 2\,(\vec{a}_{(1)}\cdot d\vec{y})\wedge dx^2\wedge dx^3 + 2\,(\vec{a}_{(2)}\cdot d\vec{y})\wedge dx^4\wedge dx^5 + 2\,(\vec{a}_{(3)}\cdot d\vec{y})\wedge dx^6\wedge dx^7$$

Arbitrary functions of y^a: $B_0, \ldots, B_4,\ \vec{\omega},\ \vec{a}_{(1)},\ \vec{a}_{(2)},\ \vec{a}_{(3)}$ ($\vec{\omega}, \vec{a}_{(1)}, \vec{a}_{(2)}, \vec{a}_{(3)}$: R^4 vectors)

Form determined by zero-force condition of mutually BPS brane probes:

- M2-branes parallel to any T^2
- M2-branes parallel to any two directions in R^4
- M5-branes parallel to one direction in each pair (2,3), (4,5), (6,7) and to any two directions in R^4

In particular:

- Fixes general form of $C^{(3)}$
- Fixes coefficients of $e^1\wedge e^2\wedge e^3$, $e^1\wedge e^4\wedge e^5$, $e^1\wedge e^6\wedge e^7$ in $C^{(3)}$
- Implies: $B_1 + B_2 + B_3 = 0$ and $B_0 = -2\,B_4$

The Ansatz

$$e^1 = e^{-2A_1-2A_2-2A_3}\left(dx^1 + \vec{\omega}\cdot d\vec{y}\right),$$
$$e^2 = e^{-2A_1+A_2+A_3}dx^2, \qquad e^3 = e^{-2A_1+A_2+A_3}dx^3,$$
$$e^4 = e^{A_1-2A_2+A_3}dx^4, \qquad e^5 = e^{A_1-2A_2+A_3}dx^5,$$
$$e^6 = e^{A_1+A_2-2A_3}dx^6, \qquad e^7 = e^{A_1+A_2-2A_3}dx^7, \qquad e^{7+a} = e^{A_1+A_2+A_3}dy^a$$

$$C^{(3)} = -\left(dx^1 + \vec{\omega}\cdot d\vec{y}\right)\wedge\left(Z_1\,dx^2\wedge dx^3 + Z_1\,dx^4\wedge dx^5 + Z_1\,dx^6\wedge dx^7\right) + \\ + 2\left(\vec{a}_{(1)}\cdot d\vec{y}\right)\wedge dx^2\wedge dx^3 + 2\left(\vec{a}_{(2)}\cdot d\vec{y}\right)\wedge dx^4\wedge dx^5 + 2\left(\vec{a}_{(3)}\cdot d\vec{y}\right)\wedge dx^6\wedge dx^7$$

- M2 branes ⟷ Functions $Z_j \equiv e^{6A_j}$
- M5 branes ⟷ Maxwell fields $F^{(j)} \equiv d\vec{a}_{(j)}$
- Rotation ⟷ Maxwell field $F^{(0)} \equiv d\vec{\omega}$

Solving the supersymmetry transformations

$\delta\psi_{2,3} = 0$

$\delta\psi_{4,5} = 0$

$\delta\psi_{5,6} = 0$

$$\left(e^{-2B_j} \not{F}^{(j)} - e^{-2B_k} \not{F}^{(k)}\right)\epsilon = 0, \quad j = 1,2,3$$

+ metric function relations

$$\delta\psi_1 = 0 \quad \Rightarrow \quad \sum_j e^{-2B_j} \not{F}^{(j)} \epsilon = 0$$

Hence $$\not{F}^{(j)} \epsilon = 0, \quad j = 1,2,3$$

Recall that on the $\mathbf{R}^4$:

$$\frac{1}{2}\left(1\!\!1 - \Gamma^{891011}\right)\varepsilon = 0$$

Therefore, on the flat $\mathbf{R}^4$ we have:

$$F^{(j)} = *F^{(j)}, \quad j = 1,2,3$$ ☆

Finally, $\delta\psi_{8,9,10,11} = 0$ yields one new equation:

$$\left(F^{(0)} + *F^{(0)}\right) = 2 \sum_j Z_j F^{(j)}$$ ☆

If the two equations ☆ are satisfied then all of the supersymmetry variations vanish

Solving the equations of motion

Important shortcut: **If $K^{\mu} \equiv \bar{\varepsilon}\Gamma^{\mu}\varepsilon$ is timelike then it suffices to check the Maxwell equations** (the Einstein equations follow automatically by integrability). **Gauntlett and Pakis, hep-th/0212008**

Basic idea: Integrability

$$\frac{1}{4} R_{\mu\nu\alpha\beta}\Gamma^{\alpha\beta}\epsilon = [\nabla_{\mu}, \nabla_{\nu}]\,\epsilon = (\nabla F)\,\epsilon + F\,F\,\epsilon$$

Take a trace, use the Maxwell equations, then what remains is ***(all)*** the Einstein equations

Complete solution to equations of motion and supersymmetry

$$F^{(j)} = *F^{(j)}, \quad j = 1,2,3$$

New equations from $d{*}F = (1/2)\, F \wedge F$

$$d * d\, Z_i = 4\, F^{(j)} \wedge F^{(k)}$$

i, j, k all distinct

$$(F^{(0)} + *F^{(0)}) = 2 \sum_j Z_j\, F^{(j)}$$

- **The system is *linear* is it is solved in this order**
- **Only need to solve Maxwell/Laplace in flat R^4**
- **Non-linearities only appear in source terms**
- **Free to pick any source distributions one wishes .....**

Ring solutions in general

First solve $F^{(j)} = *F^{(j)}$ by taking

$$\vec{b}_{(j)}(\vec{y}) \equiv \int_0^{2\pi} \frac{\sigma_j(\phi)\,\vec{\mu}_j{}'(\phi)}{|\vec{y}-\vec{\mu}_j(\phi)|^2}\,d\phi\,, \qquad F^{(j)} = (1+*)\,(d\,(\vec{b}_{(j)}\cdot d\vec{y}))$$

The M5 branes lie along curves $\vec{y} = \vec{\mu}_j(\phi)$. M5 brane charge density, $\sigma_j(\phi)$, is constant along the curve $\Rightarrow$ Dipole charges, q_j.

Simplicity (bound, single microstate) take $\vec{\mu}_j(\phi) = \vec{\mu}(\phi)$, j = 1,2,3.

Next solve $d * d\, Z_i = 4\, F^{(j)} \wedge F^{(k)}$ by taking

$$Z_i(\vec{y}) = c_i + \int \frac{\rho_i(\vec{z}) + 2|\varepsilon^{ijk}| * (F^{(j)} \wedge F^{(k)})(\vec{z})}{(\vec{y}-\vec{z})^2}\, d^4 z$$

One can choose the sources, $\rho_i(\vec{z})$, arbitrarily. Point sources yield BMPV black holes. For a bound, single microstate supertube one presumably(?) should choose the sources to lie on the ring:

$\Rightarrow$ Three more arbitrary functions, $\rho_i(\phi)$.

Finally solve $(d\vec{\omega} + *d\vec{\omega}) = 2 \sum_j Z_j F^{(j)}$ by taking d on both sides to obtain:

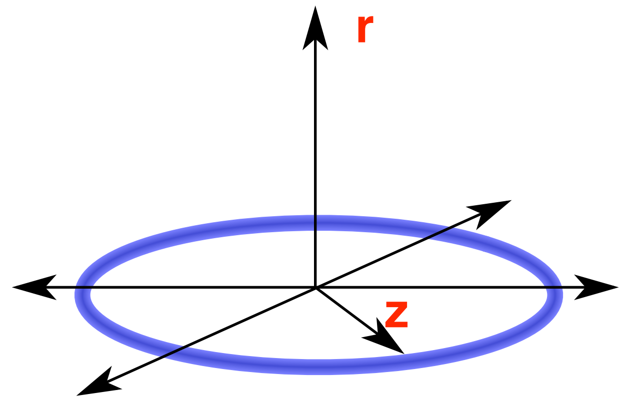
$$*d*d\vec{\omega} = *2[(dZ_1)\wedge F^{(1)} + (dZ_2)\wedge F^{(2)} + (dZ_3)\wedge F^{(3)}] \equiv J.$$

This is a standard problem in electromagnetism, with solution

$$\vec{\omega}(\vec{y}) \equiv \int_4 \frac{\vec{J}}{|\vec{y}-\vec{z}|^2}\, d^4 z$$

Homogeneous solutions and other sources restricted by absence of closed time-like curves. Rotation is limited by energy ...

The round ring



Obvious formulation: R^4 as R^2 x R^2
with polars (z, θ_1) and (r, θ_2)

Ring defined by $r = 0$, $z = R$.

Better coordinates:

$$x = -\frac{z^2 + r^2 - R^2}{\sqrt{((z-R)^2 + r^2)((z+R)^2 + r^2)}}, \quad y = -\frac{z^2 + r^2 + R^2}{\sqrt{((z-R)^2 + r^2)((z+R)^2 + r^2)}}$$

$$ds^2_{\mathbb{R}^4} = \frac{R^2}{(x-y)^2}\left(\frac{dy^2}{y^2-1} + (y^2-1)d\theta_1^2 + \frac{dx^2}{1-x^2} + (1-x^2)d\theta_2^2\right)$$

Simpler description of magnetic dipole field from ring:

$$F^{(j)} = q_j\,(dx \wedge d\theta_2 - dy \wedge d\theta_1)$$

Solution for Z_i:

$$Z_i = 1 + \frac{Q_i}{R}(x-y) - \frac{4 q_j q_k}{R^2}(x^2 - y^2)$$

boundary condition at infinity

Constant M2 brane density on ring (could be more exotic)

from $F^{(j)} \wedge F^{(k)}$ term

Rotation: $(d\omega + *d\omega) = 2 \sum_j Z_j F^{(j)}$ with $\omega = k_1 d\theta_1 + k_2 d\theta_2$

⇒ Inhomogeneous Legendre equations.

Regularity ⇒ Legendre polynomials

No closed time-like curves ⇒ Unique ω

The round ring ... continued

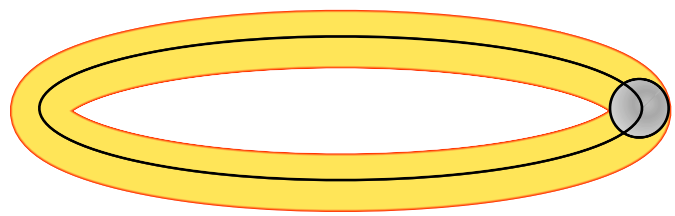
Introduce

$$A \equiv 2(q_1+q_2+q_3)\,, \quad B \equiv \frac{2}{R}(Q_1q_1+Q_2q_2+Q_3q_3)\,, \quad C \equiv -\frac{24q_1q_2q_3}{R^2}$$

then

$$k_1 = (y^2-1)\left(\frac{C}{3}(x+y) + \frac{B}{2}\right) - A(y+1)\,, \qquad k_2 = (x^2-1)\left(\frac{C}{3}(x+y) + \frac{B}{2}\right)$$

Near the ring, the warp factors blow the metric up to **S^2 x S^1**



$$ds_3^2 = \left(\frac{C^2}{9R^2}\right)^{1/3}\left[\left(\frac{9R^2}{C^2}\right)M\,d\theta_1^2 + R^2\left(d\psi^2 + \sin^2\psi\,(d\theta_1+d\theta_2)^2\right)\right]$$

Surface volume:

$$M \equiv (2q_1q_2Q_1Q_2+2q_1q_3Q_1Q_3+2q_2q_3Q_2Q_3-q_1^2Q_1^2-q_2^2Q_2^2-q_3^2Q_3^2)-\frac{1}{2}ACR^2$$

Quantized ring charges:
(L = torus length scale)

$$Q_i = \frac{\bar{N}_i l_p^6}{2L^4R}\,, \qquad q_i = \frac{n_i l_p^3}{4L^2}$$

Asymptotic charges:

$$N_1 = \bar{N}_1 + n_2n_3\,, \quad N_2 = \bar{N}_2 + n_1n_3\,, \quad N_3 = \bar{N}_3 + n_1n_2$$

$$J_1 = J^T + \frac{1}{2}\left(\sum_{i=1}^{3} n_iN_i - n_1n_2n_3\right)\,, \qquad J_2 = -\frac{1}{2}\left(\sum_{i=1}^{3} n_iN_i - n_1n_2n_3\right)$$

where $J^T = \dfrac{R^2L^4}{l_p^6}(n_1+n_2+n_3)$

Seven parameters: **q_k, Q_k, R**; Five asymptotic charges: **N_k, J_1, J_2**.

Conclusions

- General solution for the 3-charge supertube
- Simple *linear system*: Four-dimensional Maxwell/Laplace equations. Non-linearities in source terms only ...

  Good news for trying to prove Mathur's conjecture

  Huge number of solutions! Is Mathur right?
- General ring profile has seven arbitrary functions: Profile, $\vec{y} = \vec{\mu}(\phi)$, and electric densities, $\rho_i(\phi)$.

  Black hole uniqueness hugely violated ....
- Arbitrary combinations of ring profiles and charge densities are straightforward **Exotica: Variable charge density and beaded rings**
- Round ring has seven free parameters, but only five asymptotic charges.
- Supertubes are regular (except for null orbifold)
- Generalizations: Half-flat (hyperKähler) transverse spaces
- Classifying supersymmetries: A very powerful technique
- To do:
  - Find the solutions corresponding to single, bound microstates
  - Count solutions, or show that for every solution corresponding to a microstate there is a smooth supertube solution
  - IIB forms of the solutions: D1-D5-P
  - Maximum entropy calculations ....
  - Exotics: Beaded rings?, other things ....