Null-surfaces and operators with large R-charge.

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August 28, 2004

Based on:

A.M., hep-th/0311019, hep-th/0402067, hep-th/0404173, work in progress

Inspired by:

- S. Frolov, A.A. Tseytlin, "Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$", hep-th/0204226.
- A.A. Tseytlin, "Spinning strings and AdS/CFT duality", hep-th/0311139. (review)
- M. Kruczenski, "Spin chains and string theory", hep-th/0311203.
- M. Kruczenski, A. Tseytlin, "Semiclassical relativistic strings in S^5 and long coherent operators in N=4 SYM theory", hep-th/0406189.
- G. Arutyunov, M. Staudacher, "Matching Higher Conserved Charges for Strings and Spins", hep-th/0310182.

In this talk I will discuss some recent progress in AdS/CFT correspondence. The AdS/CFT correspondence is a strong-weak coupling duality. Four-dimensional $N = 4$ supersymmetric Yang-Mills theory with the coupling constant g_{YM} and 'tHooft coupling $\lambda = g_{YM}^2 N$ is conjectured to be equivalent to the Type IIB string theory on $AdS_5 \times S^5$ with the radius $R \sim \lambda^{1/4}$ and string coupling constant $g_{str} = g_{YM}^2$. This means that weakly coupled Yang-Mills (small λ) is mapped to the string theory on highly curved AdS space. When AdS space is highly curved, the string worldsheet theory becomes strongly coupled.

Therefore, the weakly coupled Yang-Mills maps to the strongly coupled string worldsheet theory, and vice versa. Nevertheless, some elements of the YM perturbation theory were recently reproduced from the string theory side. The most recent example are operators with the large R-charge.

Brief summary.

Single string states in $AdS_5 \times S^5$ correspond to single-trace operators in the $N = 4$ supersymmetric Yang-Mills theory. (We consider the large N limit.) The dynamics of the single-trace operators is described in the perturbation theory by an integrable spin chain. This spin chain has a classical continuous limit which describes a class of operators with the large R-charges. In this limit the spin chain becomes a classical continuous system. We conjecture that this classical system is equivalent to the Hamiltonian reduction of the worldsheet theory of the classical string in $AdS_5 \times S^5$ by some symmetry. The Yang-Mills perturbative expansion corresponds to considering the worldsheet of the fast moving string as a perturbation of the null-surface.

Field theory point of view.

Consider the operators of the form tr $\underbrace{\phi\phi\ldots\partial\ldots\partial\phi\ldots}_{L}$

It turns out that in some perturbative calculations the small parameter is $\frac{\lambda}{L^2}$ rather than λ. This allows to use the Yang-Mills perturbation theory in the regime where λ is large, where we can also compute on the string theory side.

It is convenient to consider instead of the single-trace operators tr $\phi\cdots\phi$ the corresponding state in the theory on $\mathbf{R}\times S^3$. This is a chain of one-particle states ("partons"). For the one-loop computations, we can consider partons as 1-particle states in the free theory. These 1-particle states form a representation of the conformal group $SO(2,4)\simeq SU(2,2)$ known as the "singleton representation". The conformal group $SU(2,2)$ together with the R-symmetry group $SU(4)$ form a bosonic part of the super-group of super-conformal transformations, called $PSU(2,2|4)$. The 1-particle states are in the "super-singleton" representation of this group.

Definition of coherent states.

Consider a special class of 1-particle states known as "coherent states". To define them, we first take the following state:

$$\psi_{\mathbf{1}} = \int_{S^3} d^3\vec{n}(\Phi_1(\vec{n}) - i\Phi_2(\vec{n}))|0> \tag{1}$$

Here Φ_1 and Φ_2 are two of the six scalar fields of the $N = 4$ super-Yang-Mills theory and $|0>$ is the conformally invariant vacuum of this theory. Our state $\psi_{\mathbf{1}}$ can be described as the creation operator of the zeroth harmonic of the field $\Phi_1 - i\Phi_2$ on S^3, acting on the vacuum.

Let us act on this state $\psi_{\mathbf{1}}$ by the superconformal group $PSU(2,2|4)$. Define ψ_g:

$$\psi_g = g.\psi_{\mathbf{1}}, \qquad g \in PSU(2,2|4) \tag{2}$$

The stabilizer of ψ_1 is $PSU(2|2) \times PSU(2|2) \times U(1)^2$, so the coherent states are parametrized by the coset space

$$\frac{PSU(2,2|4)}{PSU(2|2) \times PSU(2|2) \times U(1)^2} \tag{3}$$

This coset space has several names:

- super-Grassmanian $Gr(2|2,4|4)$
- (4,2,2) analytic superspace
- a super-symmetrization of the future tube in the complexified Minkowski space

The states ψ_g generate the whole super-singleton representation (although there are some linear relations among them). Consider chains of coherent states of partons:

$$\text{tr}\ \psi_{g_1} \otimes \psi_{g_2} \otimes \cdots \otimes \psi_{g_L} = \Psi_{[g(n)]} \tag{4}$$

Such states generate the space of L-particle states. Notice that the corresponding operators generally speaking do not have a definite engineering dimension.

Continuous limit:

- $L \to \infty$
- λ/L^2 finite but small (a small parameter of the perturbation theory),
- use $\sigma = n/L$ instead of n; $g(n)$ becomes $g(\sigma)$: a contour in $Gr(2|2, 4|4)$.

An argument for the existence of the continuous limit: If $g(\sigma) = g =$ const then $\psi_g \otimes \cdots \otimes \psi_g$ is a ferromagnetic vacuum. Continuous $g(\sigma)$ correspond to the classical long wavelength excitations about this ferromagnetic vacuum.

A calculation using the one-loop dilatation operator computed in J. A. Minahan, K. Zarembo, hep-th/0212208 and N. Beisert, M. Staudacher, hep-th/0307042. gives

$$H_{cl}^{1-loop} = \int d\sigma ||\partial_\sigma g(\sigma)||^2 \tag{5}$$

where $||dg||^2$ is the invariant metric on the coset space.

The string theory point of view.

For the classical worldsheet theory to be valid, we need at least $\lambda >> 1$. For the YM perturbation theory to work, we need $\lambda/L^2 << 1$ (this is a conjecture). Therefore, we need very large L.

Large number of partons means that the state has large R-charge, or from the point of view of the string theory the large momentum in S^5. Therefore large L corresponds to the fast moving strings.

In the limit $L = \infty$ (for fixed large λ) the string worldsheet becomes degenerate. In other words, every point on the string moves with the speed of light.

Degenerate surfaces and null surfaces.

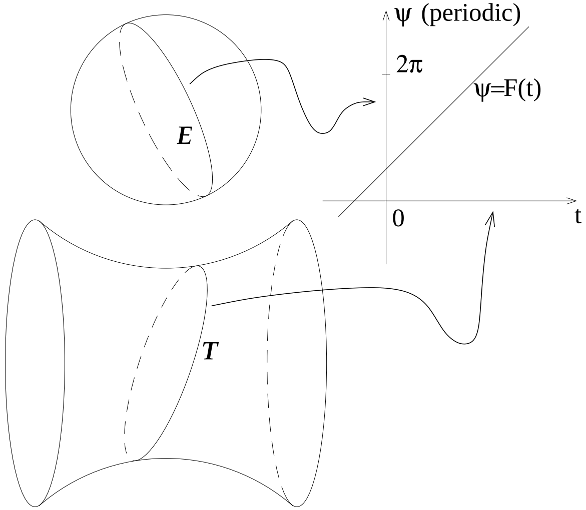
The surface is called degenerate if the induced metric is degenerate. When the string moves very fast, the worldsheet becomes a degenerate surface. The inverse is not quite true: not every degenerate surface can be obtained as a limit of a string worldsheet. Only the so-called null surfaces:

Definition. A null-surface is a degenerate surface ruled by the light rays.

There are two types of light rays in $AdS_5 \times S^5$, therefore there are two types of the null-surfaces. The first type are null-surfaces ruled by the light rays totally inside AdS_5. Such null-surfaces extend to the boundary of AdS_5. We will not discuss this type of the null-surfaces here.

What we need now is the second type of null-surfaces. They are generated by the light rays which are obtained as a diagonal in the product of a timelike geodesic in AdS_5 and an equator in S^5.

A null-geodesic of the second type in $AdS_5 \times S^5$:

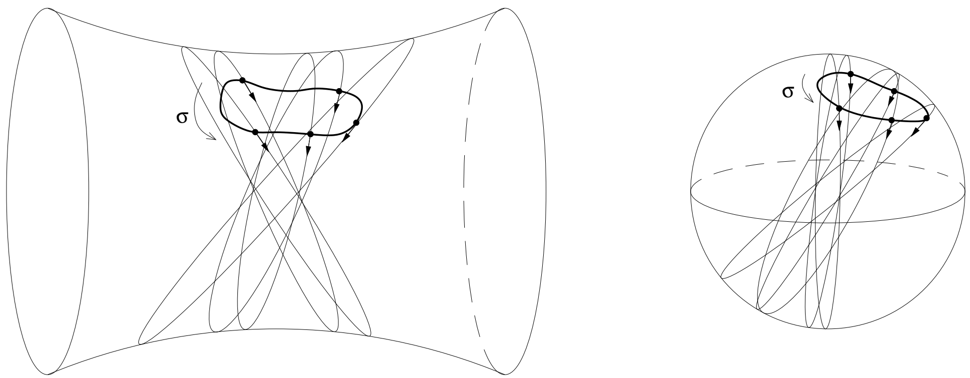


The moduli space of null-geodesics:

$$\frac{SO(2,4)}{SO(2) \times SO(4)} \times \frac{SO(6)}{SO(2) \times SO(4)} \widetilde{\times} S^1$$

This is the bosonic part of the super-Grassmanian $Gr(2|2,4|4)$ which we met on the field theory side.

A null-surface in $AdS_5 \times S^5$:



The moduli space of null-surfaces:

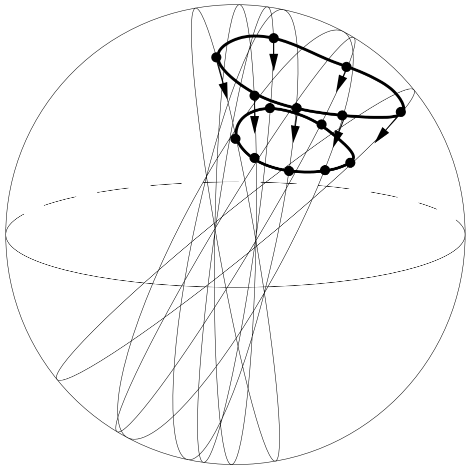
$$\frac{\text{Map}_0\left(S^1,\quad \frac{SO(2,4)}{SO(2)\times SO(4)} \times \frac{SO(6)}{SO(2)\times SO(4)}\right)\widetilde{\times}S^1}{\text{Diff}(S^1)}$$

Turning on fermionic fields on the worldsheet:

$$\frac{\text{Map}_0\left(S^1,\quad Gr(2|2,4|4)\right)\widetilde{\times}S^1}{\text{Diff}(S^1)}$$

Here $\text{Map}_0\left(S^1,\quad Gr(2|2,4|4)\right)$ is the phase space of the continuous spin chain. This means that the degrees of freedom match (modulo the fiber S^1 and Diff(S^1)) in the $\lambda/L^2 = 0$ limit. We have to explain how to extend the correspondence from null-surfaces to actual non-degenerate strings, and what happens to the fiber S^1 and to Diff(S^1).

On the null-surfaces acts $U(1)_L$, corresponding to the length of the spin chain:



For a real, non-degenerate, worldsheet of the fast moving string, can we say what is the corresponding length of the spin chain? For the continuous spin chain, the length is presumably conserved. (A related discussion: J. Minahan, "Higher Loops Beyond the SU(2) Sector", hep-th/0405243.) Therefore, we should be able to extend the action of $U(1)_L$ from the null-surfaces to the fast-moving strings.

We should find the corresponding conserved charge in the classical string worldsheet theory.

The charge-hunt

Wanted: $U(1)_L$

Description:

- Preserves the symplectic structure.
- Has periodic orbits.
- Commutes with $PSU(2,2|4)$
- In perturbation theory around the null-surface, acts locally at each order in the perturbation theory.
- Acts on the null surfaces, as described above.

Reward: The statement of equivalence of the classical spin chain and the classical string theory involves this $U(1)_L$. Namely, we conjecture that:

The phase space of the classical continuous spin chain is equivalent to the Hamiltonian reduction of the phase space of the classical string by this $U(1)_L$. This equivalence respects the action of $PSU(2,2|4)$.

$U(1)_L$ and local conserved charges.

An infinite family of local conserved currents for a string moving in $A \times S^n$ was found in: K. Pohlmeyer, Comm. Math. Phys. **46** 207-221 (1976). The first charges are:

$$Q^{[1]} = \int d\tau^+ \sqrt{\left(\frac{\partial x_S}{\partial \tau^+}, \frac{\partial x_S}{\partial \tau^+}\right)}$$

$$\widetilde{Q}^{[1]} = \int d\tau^- \sqrt{\left(\frac{\partial x_S}{\partial \tau^-}, \frac{\partial x_S}{\partial \tau^-}\right)}$$

We will argue that the generator of $U(1)_L$ is some combination of these and higher local conserved charges. The combination $Q^{[1]} - \widetilde{Q}^{[1]}$ is almost what we need. It is a conserved charge, local, acts as $U(1)_L$ on the null-surfaces. But: the orbits of the Hamiltonian $Q^{[1]} - \widetilde{Q}^{[1]}$ are not closed. (They are closed only on the null-surfaces.) The remedy is: add higher conserved charges. What is the right combination of higher conserved charges?

The work of G. Arutyunov and M. Staudacher.

In hep-th/0310182 Arutyunov and Staudacher evaluated the local conserved charges on some special solutions — rigid strings. These rigid strings depend on finitely many parameters (moduli). Corresponding field theory operators are known apriori (the local extrema of the anomalous dimension). The local conserved charges computed in hep-th/0310182 have an expansion:

$$\mathcal{E}_n = \delta_{2,n}\mathcal{J} + \frac{\epsilon_n^{(1)}}{\mathcal{J}} + \frac{\epsilon_n^{(2)}}{\mathcal{J}^3} + \frac{\epsilon_n^{(3)}}{\mathcal{J}^5} + \ldots \tag{6}$$

The coefficients $\epsilon_n^{(m)}$ depend on what kind of a rigid string is considered (the ratio of spins). But Arutyunov and Staudacher noticed that the coefficients $\epsilon_n^{(m)}$ for different values of n are not independent: they satisfy a relation, universal for all kinds of rigid solutions.

For example, up to the terms of the order $\frac{1}{\mathcal{J}^9}$:

$$\mathcal{E}_{10} + \frac{74}{7}\mathcal{E}_8 + \frac{1898}{35}\mathcal{E}_6 + \frac{6922}{35}\mathcal{E}_4 + \frac{32768}{35}(\mathcal{E}_2 - \mathcal{J}) = 0 + \mathcal{O}\left(\frac{1}{\mathcal{J}^9}\right)$$

This implies:

$$\mathcal{J} = \mathcal{E}_2 + \frac{6922}{32768}\mathcal{E}_4 + \frac{1898}{32768}\mathcal{E}_6 + \frac{370}{32768}\mathcal{E}_8 + \frac{35}{32768}\mathcal{E}_{10} + \ldots$$

Conjecture: this expression is the expansion of the generating function of $U(1)_L$ to the order $\frac{1}{|p|^9}$. We need to verify that this combination of charges generates periodic orbits. We verified it at the first nontrivial order (up to the terms of the order $1/|p|^3$).

Finite dimensional integrable systems of dimension $2n$ have n action variables — functions generating periodic orbits. Here we find an action variable $\mathcal{J}$ in an infinite-dimension system — classical string in $AdS_5 \times S^5$. This action variable corresponds to the length of the operator (the number of elementary fields under the trace) on the field theory side. Reducing by the corresponding $U(1)$ should give a system equivalent to the classical continuous spin chain. In the null-surface limit, the reduction removes the fiber S^1 and the density of $\mathcal{J}$ can be used to parametrize the null-surface (remove Diff(S^1)).