

# I=2 S-wave pion scattering phase shift with two flavor fullQCD

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(for CP-PACS Collaboration:

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Reference CP-PACS Collaboration, hep-lat/0402025

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# 1. Introduction

## Motivation

- Understanding of hadron dynamics based on lattice QCD

Strict test of Standard model requires comparison of theory and experiment. But one of main theoretical uncertainties is hadronic effect.

Especially  $\pi\pi$  scattering case, many works employed effective theory such as **chiral perturbation theory (ChPT)**.

In order to calculate the scattering based on QCD, non-perturbative method is required.



One of possible methods is lattice QCD.

- 'Dynamical' physical quantity

Most of the lattice studies have focused on 'static' physical quantities, e.g. hadron spectrum.



non-perturbative test of QCD

It is also important to study 'dynamical' physical quantities, e.g. scattering phase shift and decay width, beyond the static physical quantities.

- $I = 2$   $\pi\pi$  scattering is one of simple hadron scatterings.

Only scattering ( $\pi\pi$ ) state exists in  $I = 2$  system at low energy region.

In  $I = 0$  or 1 system, not only scattering ( $\pi\pi$ ) state but also unstable ( $\sigma$  or  $\rho$ ) states exist.

- First step toward decays of hadrons

$$I = 1 \text{ Channel} \quad \rho \rightarrow \pi\pi$$

$$I = 0 \text{ Channel} \quad \sigma \rightarrow \pi\pi$$

$$I = 0, 2 \text{ Channel} \quad K \rightarrow \pi\pi \text{ direct calculation}$$

( Traditional calculation method is  $K \rightarrow 0$  and  $K \rightarrow \pi$  relate to  $K \rightarrow \pi\pi$  with ChPT.)

## Previous work of $I = 2$ $\pi\pi$ scattering system

scattering length  $a_0 = \lim_{p \rightarrow 0} \delta_0(p)/p$

- Sharpe, Gupta, and Kilcup, 1992
- Kuramashi *et al.*, 1993
- Lin, Zhang, Chen, and Ma, 2001
- Du, Meng, Miao and Liu, 2004
- Gupta, Patel, and Sharpe, 1993
- JLQCD Collaboration, 1999
- BGR Collaboration, 2003
- Gattringer, 2004

## scattering phase shift $\delta(p)$

more difficult than  $a_0$ , so that only pioneering studies have been reported.

- Fiebig, Rabitsch, Markum, and Mihály, 2000  
 $\pi\pi$  potential
- CP-PACS Collaboration, 2002  
finite volume method in center of mass system
- Kim, 2003  
finite volume method with anti-periodic boundary condition

They employed quenched approximation, and did not take the continuum limit.

In the calculation of phase shift  $\delta(p)$

1. dynamical  $u, d$  quark effect ( $N_f = 2$  full QCD)  
get rid of systematic error of unitarity violation
2. three different lattice spacings  
take the continuum limit
3. center of mass and two laboratory systems  
add little numerical costs, more dense sampling of energy states

The aim of this work

obtain  $\delta(p)$  with two flavor full QCD in the continuum limit

## Plan of talk

1. Introduction ✓

2. Methods

- Finite volume method
- Diagonalization method

3. Parameters

4. Results

- Four-point function
- Effect of diagonalization
- Scattering length
- Scattering phase shift

5. Conclusions

## 2. Methods

### 2.1 Finite volume method

finite volume  $L^3$  with periodic boundary condition

(i) Center of mass system (total momentum  $\mathbf{P} = \mathbf{0}$ )

Lüscher, Commun. Math. Phys. 105, 153(1986)

Nucl. Phys. B354, 531(1991)

finite volume formula for S-wave ( $l = 0$ )

$$\tan \delta(\bar{p}) = \frac{\pi^{3/2} \sqrt{\bar{n}}}{Z_{00}(1; \bar{n})},$$

$$\bar{E} = 2\sqrt{m_\pi^2 + \bar{p}^2}, \quad \bar{p}^2 = \left(\frac{2\pi}{L}\right)^2 \bar{n}, \quad \bar{n} \notin Z, \quad Z_{00}(1; \bar{n}) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{n} \in \mathbf{Z}^3} \frac{1}{\mathbf{n}^2 - \bar{n}}$$

$\delta(p)$  is obtained from  $\bar{E}$ .

$\bar{p}$  is discrete due to finite volume.

(ii) Laboratory system  $\pi(\mathbf{p}_1) \longrightarrow \leftarrow \pi(\mathbf{p}_2)$  ( $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{P} \neq \mathbf{0}$ )

Rummukainen and Gottlieb, Nucl. Phys. B450, 397(1995)

General finite volume formula for S-wave ( $l = 0$ )

$$\tan \delta(\bar{p}) = \frac{\gamma \pi^{3/2} \sqrt{\bar{n}}}{Z_{00}^P(1; \bar{n})},$$

$$\bar{p}^2 = \left( \frac{\bar{E}^2}{4} - m_\pi^2 \right) = \left( \frac{2\pi}{L} \right)^2 \bar{n}, \quad \bar{E} = \gamma^{-1} \bar{E}_L, \quad \gamma = \frac{\bar{E}_L}{\sqrt{\bar{E}_L^2 - \mathbf{P}^2}}$$

$$Z_{00}^P(1; \bar{n}) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{r} \in \mathbf{P}_d} \frac{1}{r^2 - \bar{n}}, \quad \mathbf{P}_d = \{\mathbf{r} = \gamma^{-1}(\mathbf{n} + \mathbf{d}/2)\}, \quad \mathbf{d} = \frac{L\mathbf{P}}{2\pi}$$

$\delta(\bar{p})$  is obtained from  $\bar{E}_L$ .

$\bar{p}$  is discrete due to finite volume.

## 2.2 Diagonalization method

**Problem on lattice** (center of mass system)  
pion four-point function

$$\begin{aligned} G_{nn}(t) &= \langle 0 | \Omega_n^\dagger(t) \Omega_n(0) | 0 \rangle, \quad \Omega_n = \pi(\mathbf{p})\pi(-\mathbf{p}) \\ &= \sum_l |V_{ln}|^2 e^{-\bar{E}_l t}, \quad V_{ln} = \langle \bar{\Omega}_l | \Omega_n | 0 \rangle \\ &\rightarrow |V_{0n}|^2 e^{-\bar{E}_0 t}, \quad t \rightarrow \infty \end{aligned}$$

where,  $\Omega_n$  : two-pion operator with  $p^2 = (2\pi/L)^2 \cdot n$ ,  
 $\langle \bar{\Omega}_l |$  : the  $l$ -th two-pion state

Four-point function behaves as a multiexponential form.

We cannot obtain  $\bar{E}_{l \neq 0}$  by single exponential fit.

## Diagonalization of four-point function matrix

Lüscher and Wolff, Nucl. Phys. B339 222(1990)

pion four-point function matrix

$$G_{nm}(t) = \langle 0 | \Omega_n^\dagger(t) \Omega_m(0) | 0 \rangle$$

$$\begin{aligned} M(t, t_0) w_\nu &= \lambda_\nu(t, t_0) w_\nu \\ \lambda_\nu(t, t_0) &= \exp(-\bar{E}_\nu(t - t_0)) \end{aligned}$$

$$M(t, t_0) = G^{-1/2}(t_0) G(t) G^{-1/2}(t_0), \quad t_0: \text{reference point}$$

We can extract  $\bar{E}_\nu$  from  $\lambda_\nu(t, t_0)$ .

In actual calculation we need a state cut-off  $N$ , i.e.  $G(t) : N \times N$  matrix. We need to investigate  $N$  dependence of results.

The same problem exists in laboratory system calculations, but it is possible to obtain  $\bar{E}_\nu$  by diagonalization method.

### 3. Parameters

$N_f = 2$  full QCD configuration ( CP-PACS Collaboration )

- Improved action

gauge : Iwasaki action  
 fermion : clover action

- Simulation parameters  $La \sim 2.5[\text{fm}]$

| $\beta$ | $c_{SW}$ | $a^{-1}[\text{GeV}]$ | $a[\text{fm}]$ | $L^3 \cdot T$   |
|---------|----------|----------------------|----------------|-----------------|
| 1.80    | 1.60     | 0.9176(93)           | 0.2250(22)     | $12^3 \cdot 24$ |
| 1.95    | 1.53     | 1.268(13)            | 0.1555(17)     | $16^3 \cdot 32$ |
| 2.10    | 1.47     | 1.833(22)            | 0.1076(13)     | $24^3 \cdot 48$ |

$a$  is determined by  $m_\rho$

- Pion mass and number of configurations

| $\beta = 1.80$ |      |      |      |      |
|----------------|------|------|------|------|
| $m_\pi$ [GeV]  | 1.06 | 0.90 | 0.76 | 0.49 |
| # of conf.     | 645  | 520  | 725  | 405  |
| $\beta = 1.95$ |      |      |      |      |
| $m_\pi$ [GeV]  | 1.13 | 0.93 | 0.76 | 0.54 |
| # of conf.     | 595  | 690  | 685  | 495  |
| $\beta = 2.10$ |      |      |      |      |
| $m_\pi$ [GeV]  | 1.15 | 0.95 | 0.78 | 0.54 |
| # of conf.     | 395  | 390  | 380  | 640  |

Systems ( $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ ,  $\mathbf{P} : \frac{2\pi}{L}$  unit)

- Center of mass system CM
- Laboratory system 1 L1
- Laboratory system 2 L2

|    | $\mathbf{P}$ | size of $G(t)$           | state              |
|----|--------------|--------------------------|--------------------|
| CM | (0, 0, 0)    | $3 \times 3$ ( $N = 3$ ) | <u>0, 1</u> , 2    |
| L1 | (1, 0, 0)    | $4 \times 4$ ( $N = 4$ ) | <u>0, 1</u> , 2, 3 |
| L2 | (1, 1, 0)    | $4 \times 4$ ( $N = 4$ ) | <u>0, 1</u> , 2, 3 |

$G(t)$  : pion four-point function matrix,  $N$  : state cut-off

From 0, 1 states we evaluated  $\delta(p)$ .

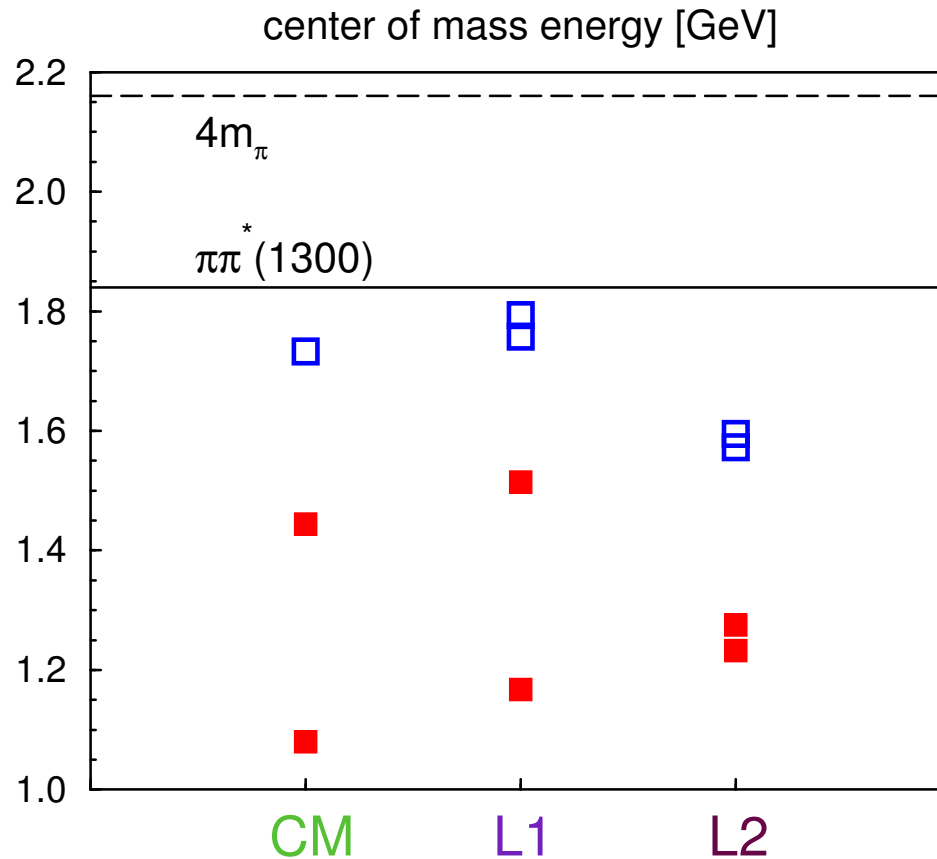
Using 2 or 3 states we investigated  $N$  dependence of 0, 1 states.

# Center of mass energy estimated in non-interacting case

at  $\beta = 2.10$ ,  $m_\pi = 0.54$  GeV

$$E = \sqrt{E_{\mathbf{P}}^2 - \mathbf{P}^2}, \quad E_{\mathbf{P}} = \sqrt{m_\pi^2 + \mathbf{p}_1^2} + \sqrt{m_\pi^2 + \mathbf{p}_2^2}$$

where  $E_{\mathbf{P}}$  is total energy for each system, and  $\mathbf{p}_i^2 = (2\pi/L)^2 \cdot n, n \in \mathbb{Z}$



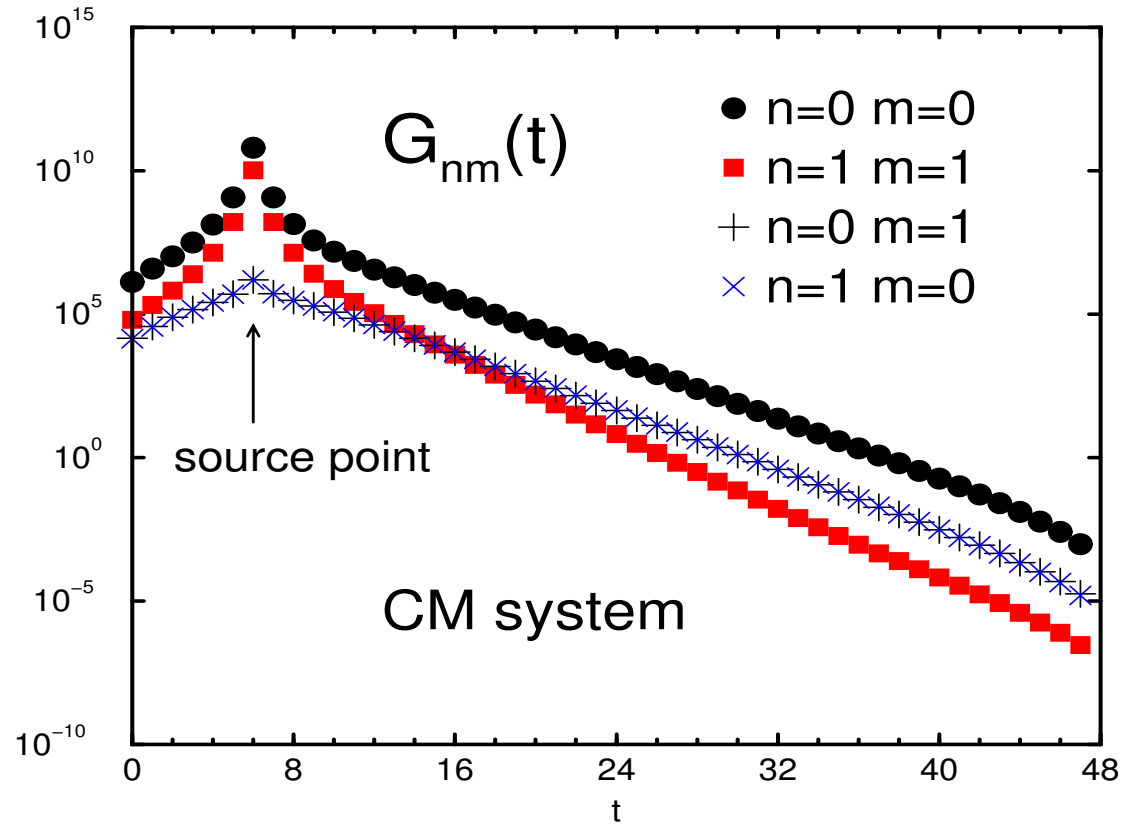
Higher states have almost same energy in L1 and L2 systems.

Measured point of phase shift is far from undesirable  $4m_\pi$  and  $\pi\pi^*$  states.

## 4. Results

### 4.1 four-point function matrix $G_{nm}(t)$

Center of mass system ( $\beta = 2.10$ ,  $m_\pi = 0.54[\text{GeV}]$ )



Cross symbols are denoted data of negative value.

Clear signal is obtained.

Off-diagonal parts are finite.

All four-point functions have same slope at large  $t$  region.

## 4.2 diagonalization

(i) Center of mass system ( $\beta = 2.10$ ,  $m_\pi = 0.54[\text{GeV}]$ )

before diagonalization (cross symbol)

$$R_n(t) = \frac{\langle 0 | \Omega_n^\dagger(t) \Omega_n(0) | 0 \rangle}{\langle 0 | \pi_n^\dagger(t) \pi_n(0) | 0 \rangle^2} \rightarrow e^{-(\bar{E}_0 - E_n)t} \quad (t \rightarrow \infty)$$

exponentially increase ( $n \neq 0$ )

after diagonalization (open symbol)

$$D_n(t) = \frac{\lambda_n(t)}{\langle 0 | \pi_n^\dagger(t) \pi_n(0) | 0 \rangle^2} \propto e^{-(\bar{E}_n - E_n)t}$$

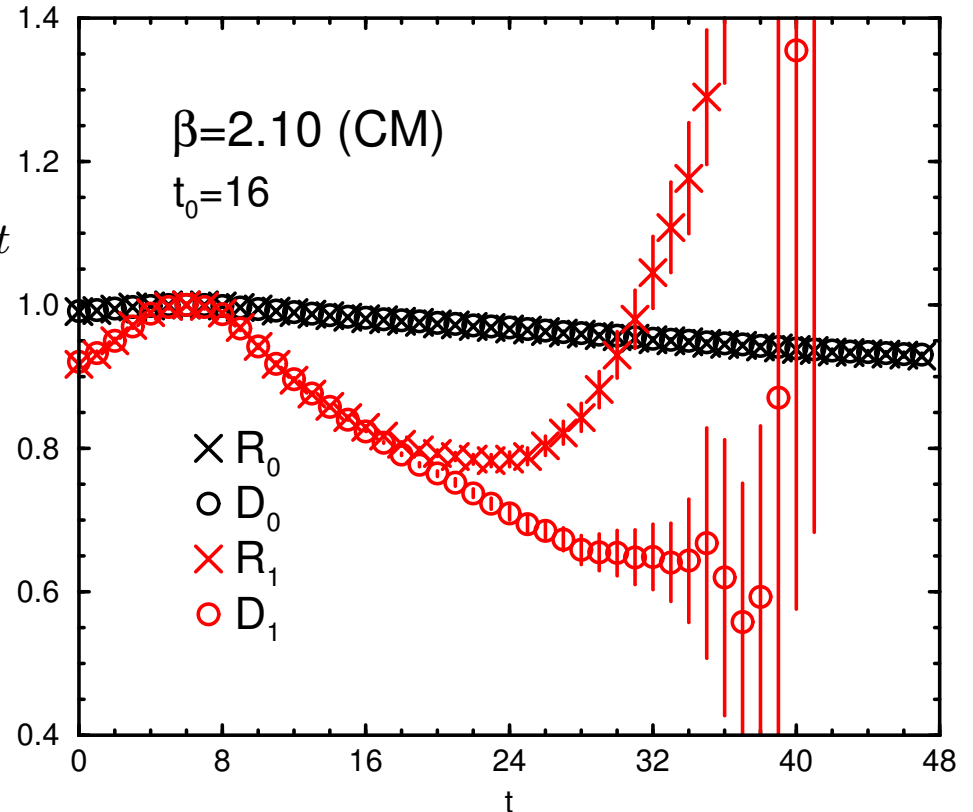
normalized by  $D_n(t_S = 6)$

$\Omega_n$  : two-pion operator

$\bar{E}_n$  : two-pion energy of  $n$ -th state with interaction

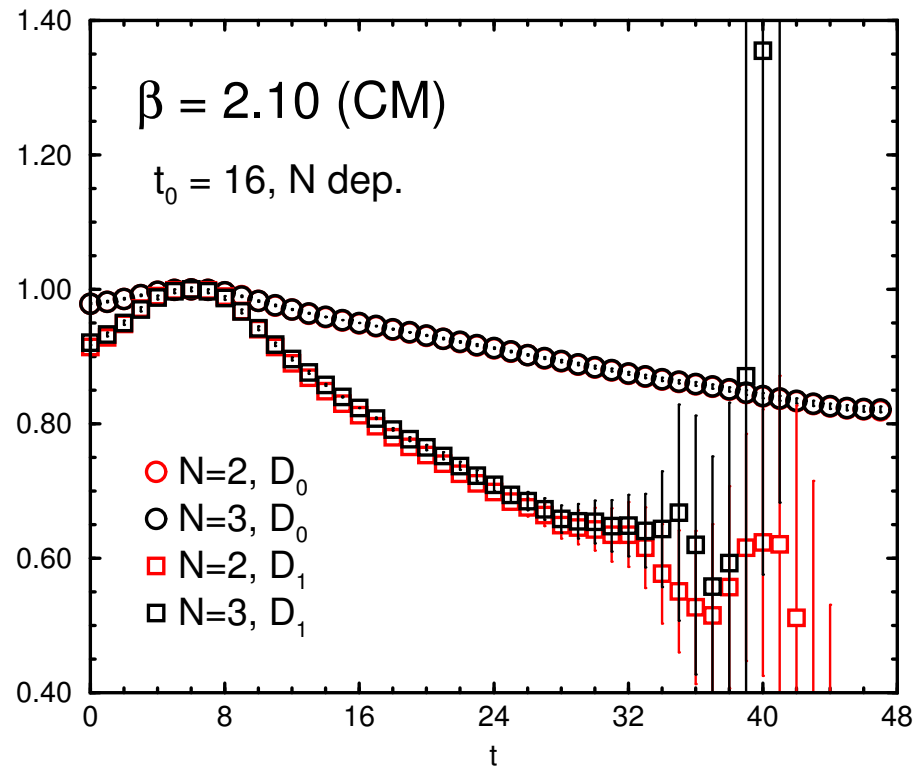
$E_n$  : two-pion energy of  $n$ -th state without interaction

$\lambda_n(t)$  :  $n$ -th state eigenvalue of  $M(t, t_0)$



We can extract  $\bar{E}_1$  from  $D_1(t)$ .

(ii) Cut-off  $N$  dependence ( $\beta = 2.10$ ,  $m_\pi = 0.54[\text{GeV}]$ )



$N$  dependence of results is reasonably small.

## 4.3 scattering length $a_0 = \lim_{\bar{p} \rightarrow 0} \tan \delta(\bar{p}) / \bar{p}$

### (i) Chiral extrapolations

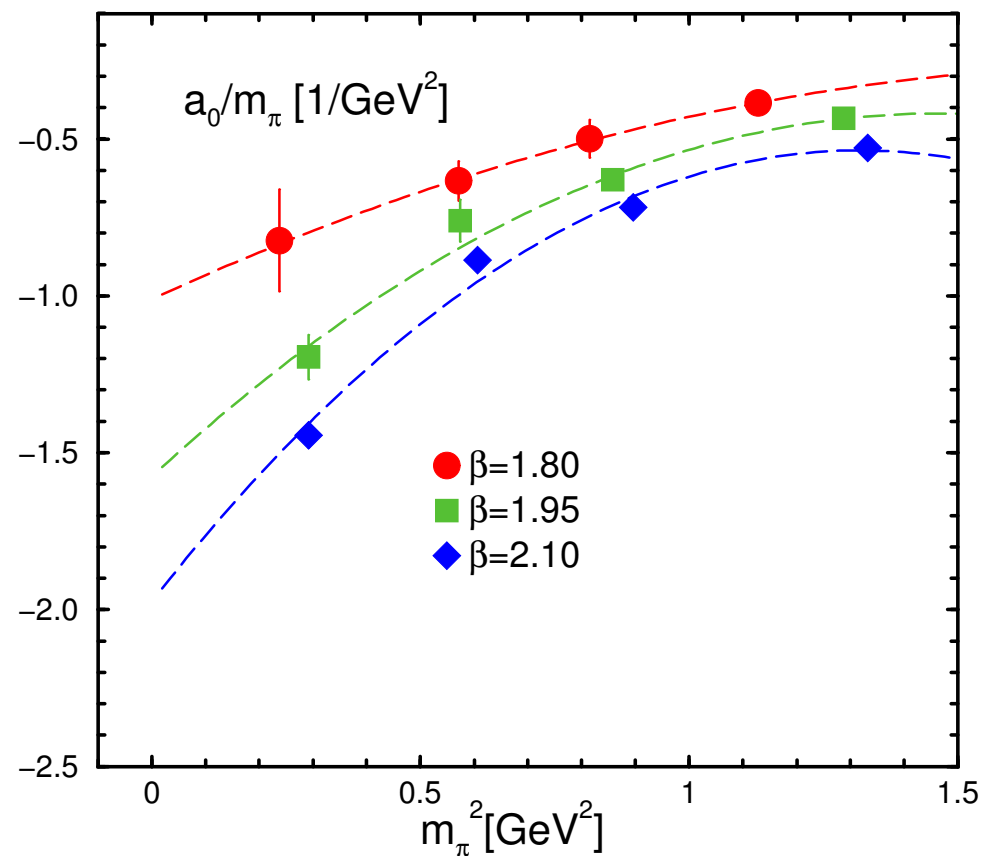
In CM system for  $n = 0$  state  $\bar{p} \approx 0$ , we obtain scattering length.

polynomial fitting form

$$a_0/m_\pi = A_{10} + A_{20}m_\pi^2 + A_{30}m_\pi^4$$

|                        |      |      |      |
|------------------------|------|------|------|
| $\beta$                | 1.80 | 1.95 | 2.10 |
| $\chi^2/\text{d.o.f.}$ | 0.02 | 2.1  | 8.2  |

$\chi^2/\text{d.o.f.}$  is very large at  $\beta = 2.10$ . Curvature causes difficulty of chiral extrapolation.



$O(a)$  and  $m_\pi$  dependence of scattering length is too large to use fitting form based on ChPT.

$$(f_{\pi}^{lat}/f_{\pi})^2 a_0/m_{\pi} [\text{GeV}^{-2}]$$

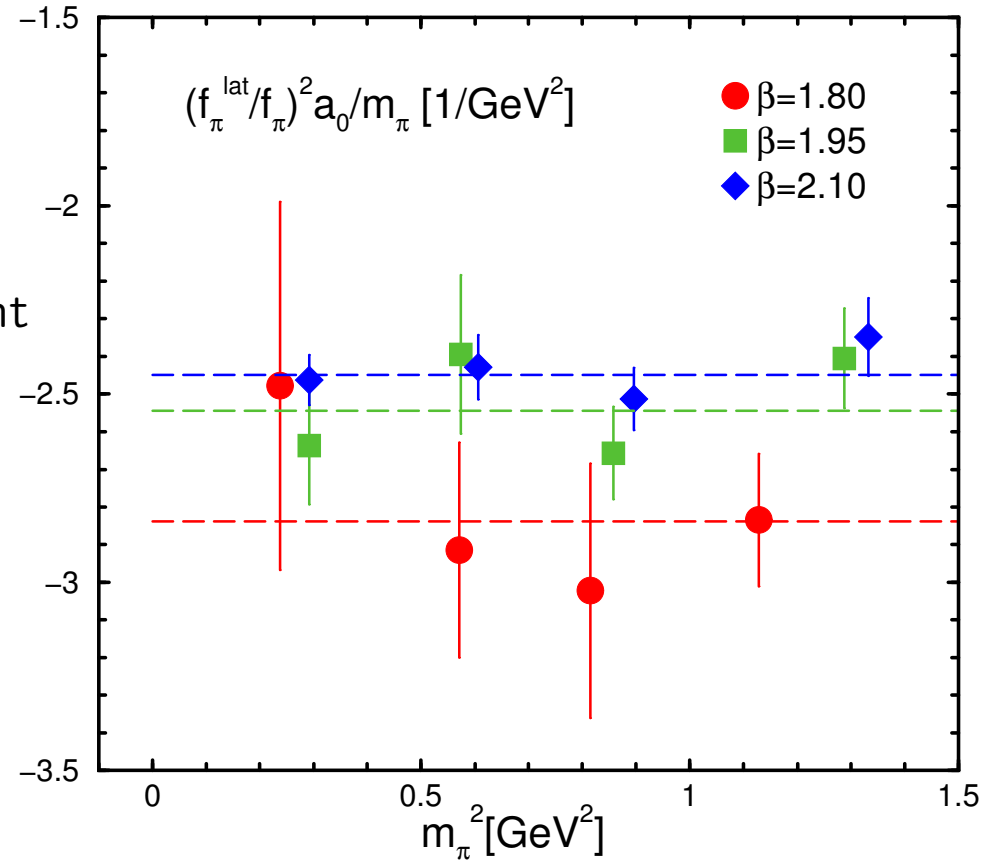
Sharpe *et al.*, Gupta *et al.*, Kuramashi *et al.*

$f_{\pi}^{lat}$  : measured pseudoscalar decay constant  
 $f_{\pi} = 93$  [MeV]

Curvatures disappear.

|                        |      |      |      |
|------------------------|------|------|------|
| $\beta$                | 1.80 | 1.95 | 2.10 |
| $\chi^2/\text{d.o.f.}$ | 0.27 | 0.94 | 0.54 |

$\chi^2/\text{d.o.f.}$  is reasonably small.



$m_{\pi}^2$  dependence of  $a_0/m_{\pi}$  is strongly correlated with one of  $(1/f_{\pi}^{lat})^2$ .

(ii) Continuum extrapolations at  $m_\pi = 0.14$  GeV

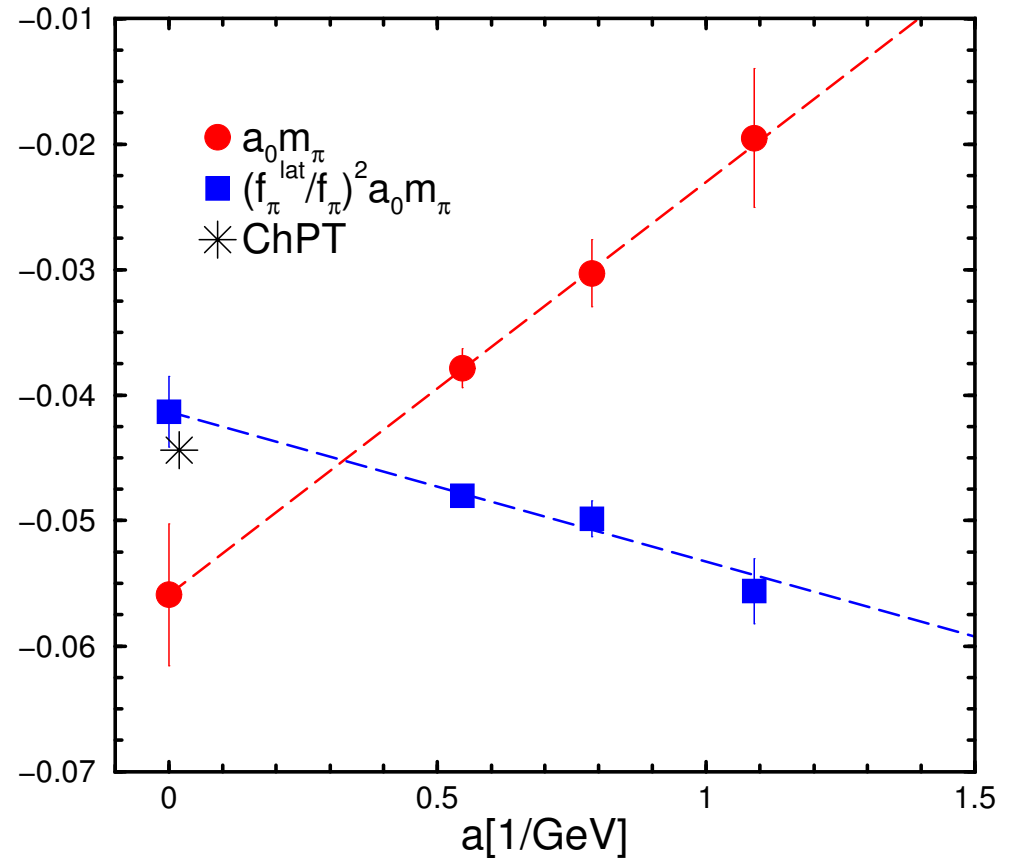
Large  $O(a)$  dependence exists.

$(f_\pi^{lat}/f_\pi)^2 a_0 m_\pi$  has a problem in the continuum limit.

$f_\pi^{lat}$  in the continuum limit seems to be smaller than  $f_\pi = 93$  MeV.

We cannot extract a reliable  $a_0$  from  $(f_\pi^{lat}/f_\pi)^2 a_0 m_\pi$ .

$a_0 m_\pi$  in the continuum limit



$$\begin{aligned}
 a_0 m_\pi &= -0.0558(56) \\
 (f_\pi^{lat}/f_\pi)^2 a_0 m_\pi &= -0.0413(28) \\
 a_0 m_\pi(\text{ChPT}) &= -0.0444(10)
 \end{aligned}$$

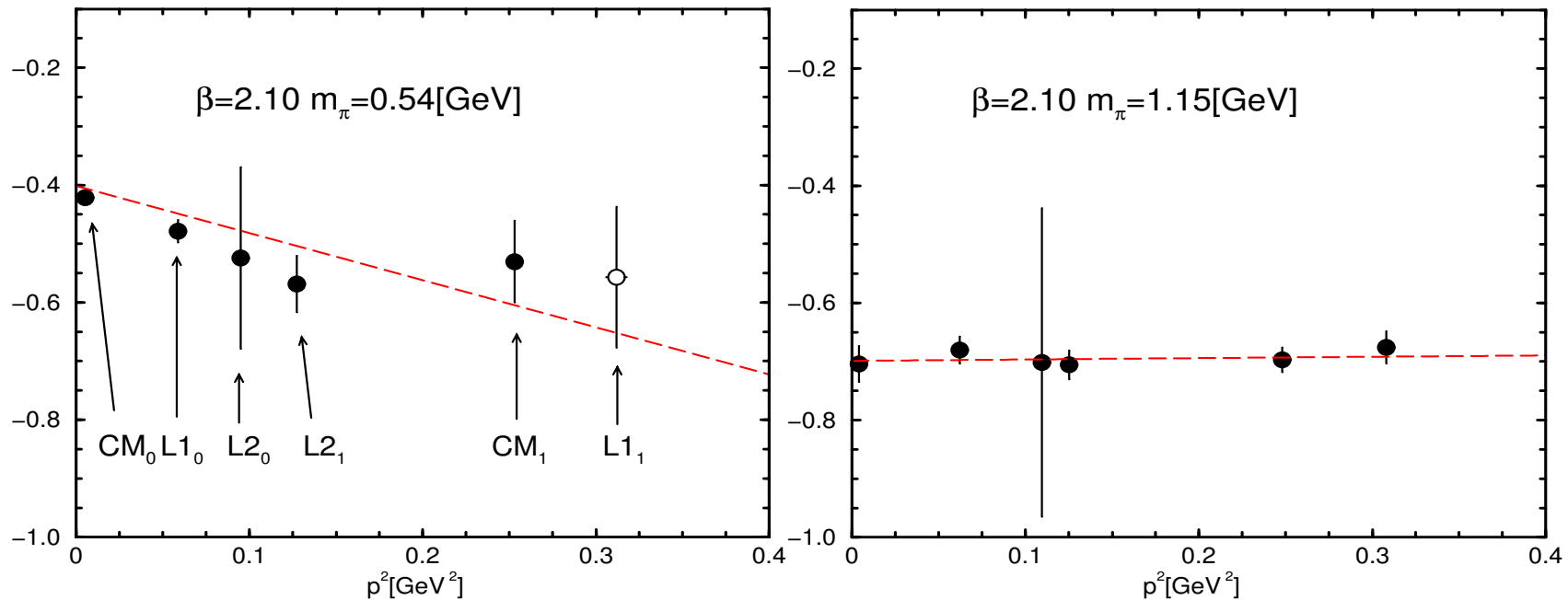
## 4.3 Scattering phase shift $\delta(\bar{p})$

### (i) Chiral extrapolations

'Scattering amplitude'  $A(\bar{p}, m_\pi) = \frac{\tan \delta(\bar{p})}{\bar{p}} \cdot \frac{\bar{E}}{2}$ ,  $\lim_{\bar{p} \rightarrow 0} A(\bar{p}, m_\pi) = a_0 m_\pi$

global fit for  $m_\pi^2$  and  $\bar{p}^2$  at each  $\beta$

$$A(\bar{p}, m_\pi) = A_{10} m_\pi^2 + A_{20} m_\pi^4 + A_{30} m_\pi^6 + A_{01} \bar{p}^2 + A_{11} m_\pi^2 \bar{p}^2 + A_{21} m_\pi^4 \bar{p}^2$$



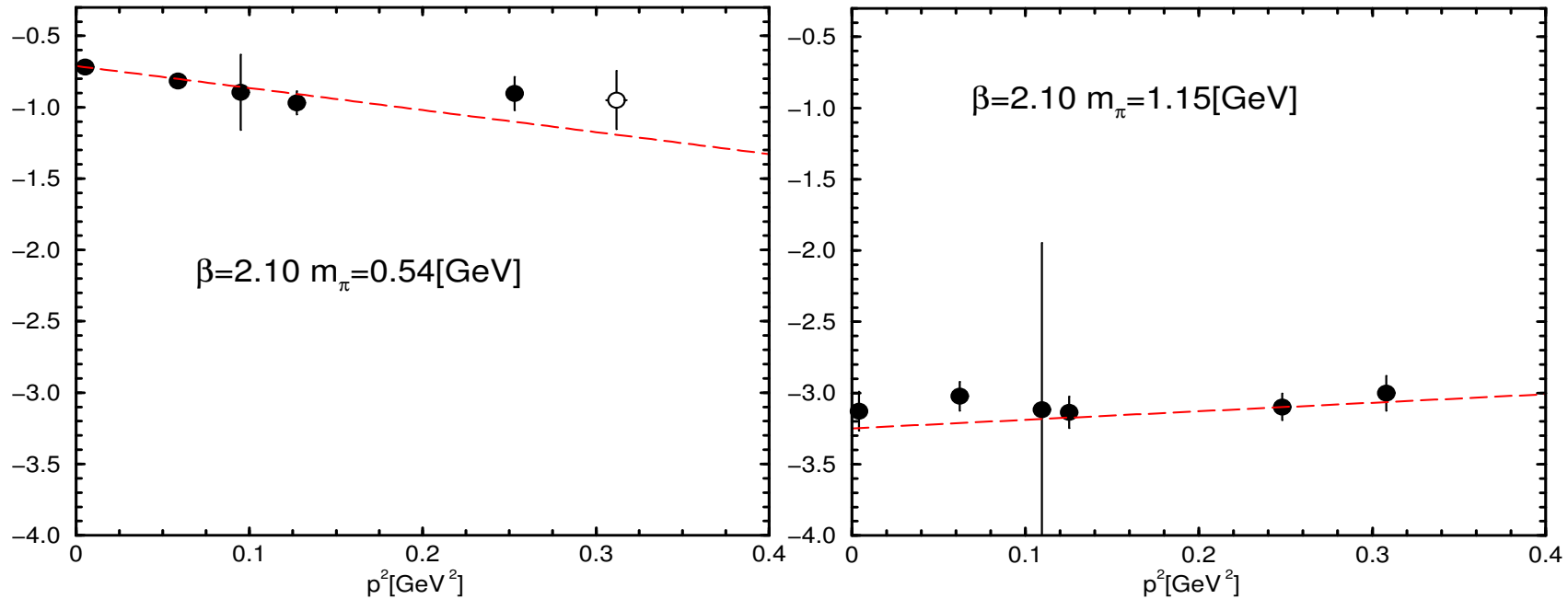
Reasonable fittings are possible with the amplitudes obtained from different systems.

|                        |      |      |      |
|------------------------|------|------|------|
| $\beta$                | 1.80 | 1.95 | 2.10 |
| $\chi^2/\text{d.o.f.}$ | 0.90 | 0.64 | 1.33 |

Normalized amplitude  $A^f(\bar{p}, m_\pi) = (f_\pi^{lat}/f_\pi)^2 A(\bar{p}, m_\pi)$

global fit for  $m_\pi^2$  and  $\bar{p}^2$  at each  $\beta$

$$A^f(\bar{p}, m_\pi) = A_{10}^f m_\pi^2 + A_{01}^f \bar{p}^2 + A_{11}^f m_\pi^2 \bar{p}^2$$

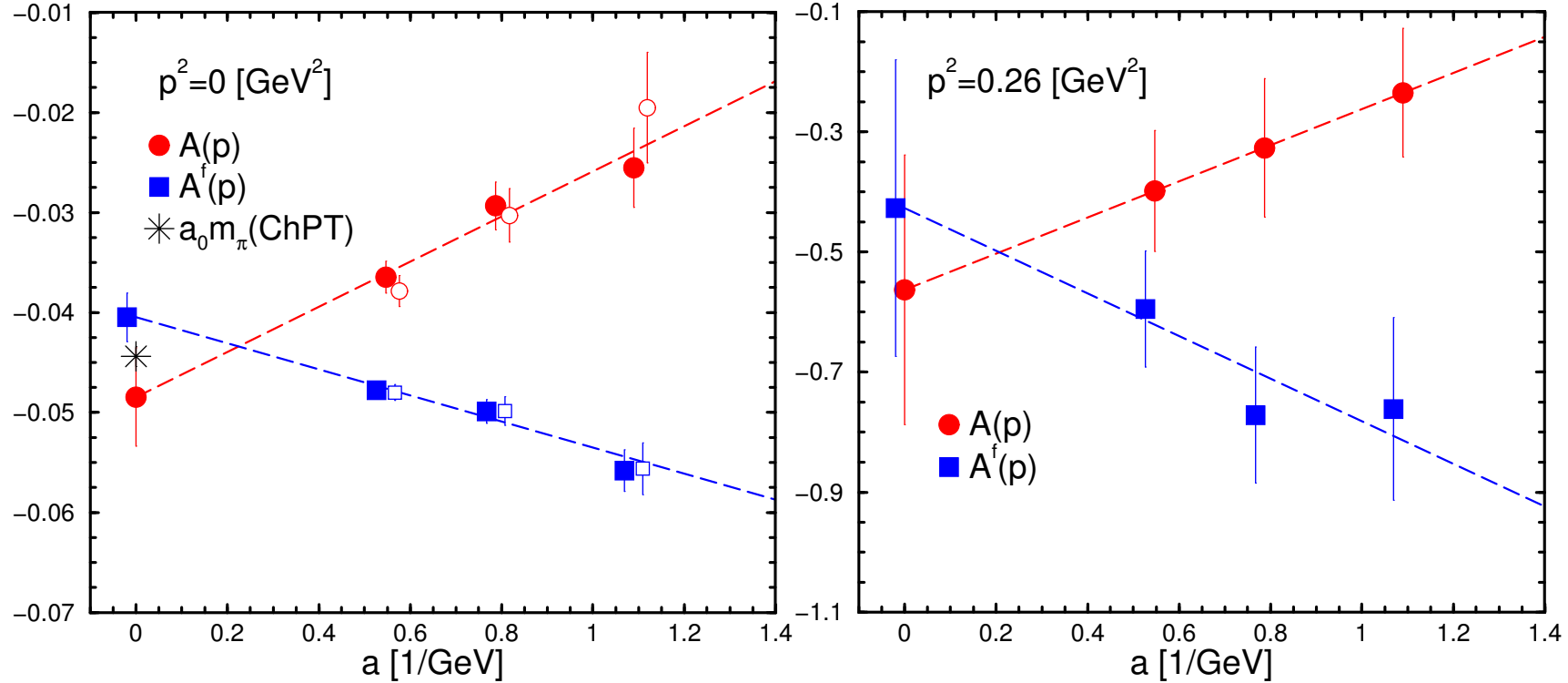


Fittings are also reasonable with the amplitudes obtained from different systems.

|                        |      |      |      |
|------------------------|------|------|------|
| $\beta$                | 1.80 | 1.95 | 2.10 |
| $\chi^2/\text{d.o.f.}$ | 0.77 | 0.50 | 0.75 |

(ii) Continuum extrapolations at  $m_\pi = 0.14[\text{GeV}]$

linear fit at fixed  $\bar{p}$  :  $A(\bar{p}) + aA^{(a)}(\bar{p})$



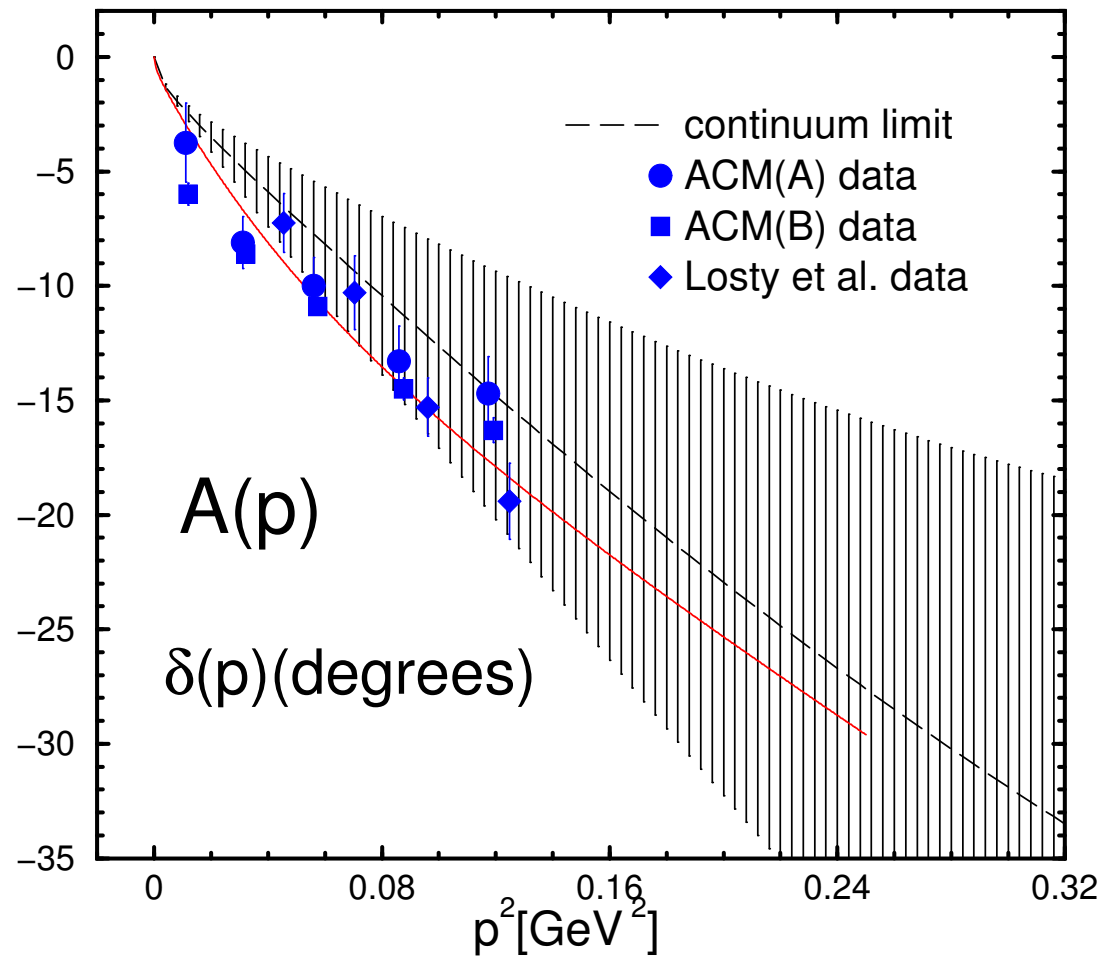
$$A(0) = -0.0484(49), \quad A^f(0) = -0.0404(24)$$

$$a_0 m_\pi (\text{ChPT}) = -0.0444(10)$$

Large  $O(a)$  effect exists.

Difference of the results decreases as lattice spacing decreases.

$\delta(p)$  (degrees) in  $a = 0$  at  $m_\pi = 0.14[\text{GeV}]$



Symbols are the experimental data. ( Hoogland *et al.*, Losty *et al.* )

Solid line is parametrized by experimental input. ( Colangelo *et al.* )

Result is consistent with experimental data.

## 5. Conclusions

We calculate  $I = 2$  S-wave  $\pi\pi$  scattering phase shift.

- $N_f = 2$  full QCD
- Continuum limit
- Center of mass system and two laboratory systems

Scattering length  $a_0$

- Curvature causes difficulty of chiral extrapolation.
- Large  $O(a)$  effect exists.

$$\begin{aligned} a_0 m_\pi &= -0.0558(56) \\ (f_\pi^{lat}/f_\pi)^2 a_0 m_\pi &= -0.0413(28) \end{aligned}$$

Scattering phase shift  $\delta(p)$

- Large  $O(a)$  effect exists.
- Result in the continuum limit is consistent with experimental data.

$$\begin{aligned} a_0 m_\pi &= -0.0484(49) \\ (f_\pi^{lat}/f_\pi)^2 a_0 m_\pi &= -0.0404(24) \end{aligned}$$

## future work

1. Calculation at smaller  $m_\pi$  and  $a$   
Chiral and continuum extrapolations
2. Volume dependence  
method with  $\pi\pi$  wave function
3.  $I = 1$   $\pi\pi$  resonance scattering phase shift  
 $\rho \rightarrow \pi\pi$  decay
4.  $I = 0$   $\pi\pi$  scattering phase shift
5.  $K \rightarrow \pi\pi$  decay