Collinear and soft resummation in the large-x limit

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- Soft logarithms
- Collinear logarithms
- Resummed cross section
- NNLO corrections
- NNNLO corrections

Calculation of cross sections in perturbative QCD: factorization of long-distance and short-distance physics

$$\sigma = \sum_f \int \left[\prod_i dx_i \, \phi_{f/h_i}(x_i, \mu_F^2) \right] \hat{\sigma}(s, t_i, \mu_F, \mu_R)$$

Hard-scattering factors - perturbatively calculable

Parton distributions - determined from experiment

Near threshold for production of the system restricted phase space for real gluon emission

Incomplete cancellation of infrared divergences between real and virtual graphs $\rightarrow$ large logarithms

Soft and collinear corrections $\longrightarrow$ plus distributions

1PI kinematics, $s_4 = s + t + u - \sum m^2 \rightarrow 0$ at threshold

$$\mathcal{D}_l(s_4) \equiv \left[\frac{\ln^l(s_4/M^2)}{s_4} \right]_+$$

PIM kinematics, $x = Q^2/s \rightarrow 1$ at threshold

$$\mathcal{D}_l(x) \equiv \left[\frac{\ln^l(1-x)}{1-x} \right]_+$$

Collinear corrections 1PI: $\ln^l(s_4/M^2)$ PIM: $\ln^l(1-x)$

Define moments of the cross section

$\hat{\sigma}(N) = \int dx \, x^{N-1} \hat{\sigma}(x)$

Soft corrections

$\left[\frac{\ln^{2n-1}(1-x)}{1-x}\right]_+ \rightarrow \ln^{2n} N$

We can formally resum these logarithms to all orders in α_s: factorize soft gluons from the hard scattering

N. Kidonakis and G. Sterman, Nucl. Phys. B 505, 321 (1997)

Invert back to momentum space

Let x_{th} denote s_4/M^2 or $1-x$

At NLO, $\mathcal{D}_1(x_{th})$, $\mathcal{D}_0(x_{th})$, and $\delta(x_{th})$ terms

At NNLO, $\mathcal{D}_3(x_{th})$, $\mathcal{D}_2(x_{th})$, $\mathcal{D}_1(x_{th})$, $\mathcal{D}_0(x_{th})$, and $\delta(x_{th})$ terms LL, NLL, NNLL, NNNLL

A unified approach and a master formula for calculating these distributions at NNLO for any process

N. Kidonakis, Int. J. Mod. Phys. A 19, 1793 (2004)

Collinear corrections

$\ln^n(s_4/M^2) \to \ln^n N/N$

At NLO, $\ln(x_{th})$ and constant terms

At NNLO, $\ln^3(x_{th})$, $\ln^2(x_{th})$, $\ln(x_{th})$, and constant terms

Soft and collinear formalism applied to

- Hadron-hadron and lepton-hadron colliders
- Total and differential cross sections
- 1PI and PIM kinematics
- Simple and complex color flows
- $\overline{\text{MS}}$ and DIS factorization schemes

Soft and collinear resummation

A unified approach

$$\hat{\sigma}^{res}(N) = \exp\left[\sum_i E_i(N_i) + E_i^{coll}(N_i)\right] \exp\left[\sum_j E'_j(N_j) + E'^{coll}_j(N_j)\right]$$

$$\times \exp\left[\sum_i 2\int_{\mu_F}^{\sqrt{s}} \frac{d\mu'}{\mu'}\left(\frac{\alpha_s(\mu'^2)}{\pi}\gamma_i^{(1)} + \gamma'_{i/i}\left(\alpha_s(\mu'^2)\right)\right)\right]$$

$$\times \exp\left[2\, d_{\alpha_s}\int_{\mu_R}^{\sqrt{s}} \frac{d\mu'}{\mu'}\beta\left(\alpha_s(\mu'^2)\right)\right]$$

$$\times \mathrm{Tr}\left\{H\left(\alpha_s(\mu_R^2)\right)\, \exp\left[\int_{\sqrt{s}}^{\sqrt{s}/\tilde{N}_j} \frac{d\mu'}{\mu'}\Gamma'^{\dagger}_S\left(\alpha_s(\mu'^2)\right)\right]\right.$$

$$\left.\times\, \tilde{S}\left(\alpha_s(s/\tilde{N}_j^2)\right)\, \exp\left[\int_{\sqrt{s}}^{\sqrt{s}/\tilde{N}_j} \frac{d\mu'}{\mu'}\,\Gamma'_S\left(\alpha_s(\mu'^2)\right)\right]\right\}$$

with ($\overline{\text{MS}}$ scheme)

$$E_i(N_i) = -\int_0^1 dz \frac{z^{N_i-1}-1}{1-z}\left\{\int_{(1-z)^2 s}^{\mu_F^2} \frac{d\mu'^2}{\mu'^2} A_i\left(\alpha_s(\mu'^2)\right) + \nu_i\left[\alpha_s((1-z)^2 s)\right]\right\}$$

with $A_i(\alpha_s) = C_i\left[\alpha_s/\pi + (\alpha_s/\pi)^2 K/2\right] + A_i^{(3)} + \cdots$
$\nu_i = (\alpha_s/\pi)C_i + (\alpha_s/\pi)^2\nu_i^{(2)} + \cdots$

and (for any massless final-state partons at lowest order)

$$E'_j(N_j) = \int_0^1 dz \frac{z^{N_j-1}-1}{1-z}\left\{\int_{(1-z)^2}^{1-z} \frac{d\lambda}{\lambda} A_j\left(\alpha_s(\lambda s)\right) - B'_j\left[\alpha_s((1-z)s)\right] - \nu_j\left[\alpha_s((1-z)^2 s)\right]\right\}$$

where $B'_j = (\alpha_s/\pi)B'^{(1)}_j + (\alpha_s/\pi)^2 B'^{(2)}_j + \cdots$

with $B'^{(1)}_q = 3C_F/4$ and $B'^{(1)}_g = \beta_0/4$

Leading collinear exponents

$$E_i^{coll}(N_i) \quad = \quad \int_0^1 dz \; z^{N_i-1} \left\{ \int_{(1-z)^2 s}^{\mu_F^2} \frac{d\mu'^2}{\mu'^2} A_i \left(\alpha_s(\mu'^2) \right) \right\}$$

and

$$E'^{coll}_j(N_j) \quad = \quad - \int_0^1 dz \; z^{N_j-1} \left\{ \int_{(1-z)^2}^{1-z} \frac{d\lambda}{\lambda} A_j \left(\alpha_s \left(\lambda s \right) \right) \right\}$$

γ_i are parton anomalous dimensions

H are hard scattering matrices

S are soft matrices (noncollinear soft-gluon emission)

Γ_S are soft anomalous dimension matrices -evolution of color exchange in QCD hard scattering

NLO master formula

$$\hat{\sigma}^{(1)} = \sigma^B \frac{\alpha_s(\mu_R^2)}{\pi} \{c_3 \, \mathcal{D}_1(x_{th}) + c_2 \, \mathcal{D}_0(x_{th}) + c_1 \, \delta(x_{th})\} + \frac{\alpha_s^{d_{\alpha_s}+1}(\mu_R^2)}{\pi} [A^c \, \mathcal{D}_0(x_{th}) + T_1^c \, \delta(x_{th})]$$

with

$$c_3 = \sum_i 2 \, C_i - \sum_j C_j$$

For quarks $C_F = (N_c^2 - 1)/(2N_c)$ For gluons $C_A = N_c$

$$c_2 = -\sum_i \left[C_i + 2 \, C_i \, \ln\left(\frac{-t_i}{M^2}\right) + C_i \, \ln\left(\frac{\mu_F^2}{s}\right)\right] - \sum_j \left[B'^{(1)}_j + C_j + C_j \, \ln\left(\frac{M^2}{s}\right)\right]$$

$$A^c = \mathrm{tr}\left(H^{(0)} \Gamma'^{(1)\,\dagger}_S S^{(0)} + H^{(0)} S^{(0)} \Gamma'^{(1)}_S\right) .$$

Also $c_1 = c_1^\mu + T_1$, with

$$c_1^\mu = \sum_i \left[C_i \, \ln\left(\frac{-t_i}{M^2}\right) - \gamma_i^{(1)}\right] \ln\left(\frac{\mu_F^2}{s}\right) + d_{\alpha_s} \frac{\beta_0}{4} \ln\left(\frac{\mu_R^2}{s}\right)$$

For quarks $B'^{(1)}_q = \gamma_q^{(1)} = 3C_F/4$ For gluons $B'^{(1)}_g = \gamma_g^{(1)} = \beta_0/4$

Example: The Drell-Yan process $q\bar{q} \to V$

$$\hat{\sigma}^{(1)} = \sigma^B \frac{\alpha_s(\mu_R^2)}{\pi} \left\{4C_F \left[\frac{\ln(1-x)}{1-x}\right]_+ - 2C_F \ln\left(\frac{\mu_F^2}{Q^2}\right) \left[\frac{1}{1-x}\right]_+ + c_1 \, \delta(1-x)\right\}$$

Plus collinear terms

$$\sigma^B \frac{\alpha_s(\mu_R^2)}{\pi} \left\{-4C_F \ln(1-x) + 2C_F \ln\left(\frac{\mu_F^2}{Q^2}\right)\right\}$$

NNLO master formula

$$
\begin{aligned}
\hat{\sigma}^{(2)} &= \sigma^B \frac{\alpha_s^2(\mu_R^2)}{\pi^2}\, \frac{1}{2}\, c_3^2\, \mathcal{D}_3(x_{th}) \\
&+ \sigma^B \frac{\alpha_s^2(\mu_R^2)}{\pi^2} \left\{ \frac{3}{2} c_3\, c_2 - \frac{\beta_0}{4} c_3 + \sum_j C_j \frac{\beta_0}{8} \right\} \mathcal{D}_2(x_{th}) + \frac{\alpha_s^{d_{\alpha_s}+2}(\mu_R^2)}{\pi^2}\, \frac{3}{2} c_3\, A^c\, \mathcal{D}_2(x_{th}) \\
&+ \sigma^B \frac{\alpha_s^2(\mu_R^2)}{\pi^2} \left\{ c_3\, c_1 + c_2^2 - \zeta_2\, c_3^2 - \frac{\beta_0}{2} T_2 + \frac{\beta_0}{4} c_3 \ln\left(\frac{\mu_R^2}{s}\right) + \sum_i C_i\, K \right. \\
&\quad \left. + \sum_j C_j \left[-\frac{K}{2} + \frac{\beta_0}{4} \ln\left(\frac{M^2}{s}\right) \right] - \sum_j \frac{\beta_0}{4} B'^{(1)}_j \right\} \mathcal{D}_1(x_{th}) \\
&+ \frac{\alpha_s^{d_{\alpha_s}+2}(\mu_R^2)}{\pi^2} \left\{ \left(2\, c_2 - \frac{\beta_0}{2} \right) A^c + c_3\, T_1^c + F^c \right\} \mathcal{D}_1(x_{th}) \\
&+ \sigma^B \frac{\alpha_s^2(\mu_R^2)}{\pi^2} \left\{ c_2\, c_1 - \zeta_2\, c_3\, c_2 + \zeta_3\, c_3^2 - \frac{\beta_0}{2} T_1 + \frac{\beta_0}{4} c_2 \ln\left(\frac{\mu_R^2}{s}\right) - \sum_i \nu_i^{(2)} \right. \\
&\quad + \sum_i C_i \left[\frac{\beta_0}{8} \ln^2\left(\frac{\mu_F^2}{s}\right) - \frac{K}{2} \ln\left(\frac{\mu_F^2}{s}\right) - K \ln\left(\frac{-t_i}{M^2}\right) \right] - \sum_j \left(B'^{(2)}_j + \nu_j^{(2)} \right) \\
&\quad \left. + \sum_j C_j \left[\frac{\beta_0}{8} \ln^2\left(\frac{M^2}{s}\right) - \frac{K}{2} \ln\left(\frac{M^2}{s}\right) \right] - \sum_j \frac{\beta_0}{4} B'^{(1)}_j \ln\left(\frac{M^2}{s}\right) \right\} \mathcal{D}_0(x_{th}) \\
&+ \frac{\alpha_s^{d_{\alpha_s}+2}(\mu_R^2)}{\pi^2} \left\{ \left[c_1 - \zeta_2\, c_3 + \frac{\beta_0}{4} \ln\left(\frac{\mu_R^2}{s}\right) \right] A^c + \left(c_2 - \frac{\beta_0}{2} \right) T_1^c \right. \\
&\quad \left. + F^c \ln\left(\frac{M^2}{s}\right) + G^c \right\} \mathcal{D}_0(x_{th}) + R_c^{(2)}\, \delta(x_{th})
\end{aligned}
$$

Here

$$
F^c = \mathrm{tr}\left[H^{(0)} \left(\Gamma'^{(1)\,\dagger}_S \right)^2 S^{(0)} + H^{(0)} S^{(0)} \left(\Gamma'^{(1)}_S \right)^2 + 2 H^{(0)} \Gamma'^{(1)\,\dagger}_S S^{(0)} \Gamma'^{(1)}_S \right]
$$

$$
\begin{aligned}
G^c \;=\; \mathrm{tr}\Big[& H^{(1)} \Gamma'^{(1)\,\dagger}_S S^{(0)} + H^{(1)} S^{(0)} \Gamma'^{(1)}_S + H^{(0)} \Gamma'^{(1)\,\dagger}_S S^{(1)} + H^{(0)} S^{(1)} \Gamma'^{(1)}_S \\
& + H^{(0)} \Gamma'^{(2)\,\dagger}_S S^{(0)} + H^{(0)} S^{(0)} \Gamma'^{(2)}_S \Big]
\end{aligned}
$$

NNNLO master formula

$$\hat{\sigma}^{(3)} = \sigma^B \frac{\alpha_s^3(\mu_R^2)}{\pi^3}\, \frac{1}{8}\, c_3^3\, \mathcal{D}_5(x_{th})$$

$$+ \sigma^B \frac{\alpha_s^3(\mu_R^2)}{\pi^3} \left\{\frac{5}{8} c_3^2\, c_2 - \frac{5}{2} c_3\, X_3\right\} \mathcal{D}_4(x_{th}) + \frac{\alpha_s^{d_{\alpha_s}+3}(\mu_R^2)}{\pi^3}\, \frac{5}{8}\, c_3^2\, A^c\, \mathcal{D}_4(x_{th})$$

$$+ \sigma^B \frac{\alpha_s^3(\mu_R^2)}{\pi^3} \left\{c_3\, c_2^2 + \frac{c_3^2}{2} c_1 - \zeta_2\, c_3^3 + (\beta_0 - 4c_2)\, X_3 + 2c_3\, X_2 - \sum_j C_j\, \frac{\beta_0^2}{48}\right\} \mathcal{D}_3(x_{th})$$

$$+ \frac{\alpha_s^{d_{\alpha_s}+3}(\mu_R^2)}{\pi^3} \left\{\frac{1}{2} c_3^2\, T_1^c + \left[2\, c_3\, c_2 - \frac{\beta_0}{2} c_3 - 4\, X_3\right] A^c + c_3\, F^c\right\} \mathcal{D}_3(x_{th})$$

$$+ \sigma^B \frac{\alpha_s^3(\mu_R^2)}{\pi^3} \left\{\frac{3}{2} c_3\, c_2\, c_1 + \frac{1}{2} c_2^3 - 3\, \zeta_2\, c_3^2\, c_2 + \frac{5}{2} \zeta_3\, c_3^3 + \left(-3\, c_1 + \frac{27}{2} \zeta_2\, c_3\right) X_3\right.$$

$$+ (3\, c_2 - \beta_0)\, X_2 - \frac{3}{2} c_3\, X_1 - \sum_i C_i\, \frac{\beta_1}{8} + \sum_j \frac{\beta_0^2}{16} B'^{(1)}_j + \sum_j \frac{3}{32} C_j\, \beta_1$$

$$\left. + \sum_j C_j\, \frac{\beta_0}{16} \left[\beta_0 \ln\left(\frac{\mu_R^2}{s}\right) + 2\, K - \beta_0 \ln\left(\frac{M^2}{s}\right)\right]\right\} \mathcal{D}_2(x_{th})$$

$$+ \frac{\alpha_s^{d_{\alpha_s}+3}(\mu_R^2)}{\pi^3} \left\{\left(\frac{3}{2} c_3\, c_2 - 3\, X_3\right) T_1^c + \frac{3}{2}\left[c_2 + c_3 \ln\left(\frac{M^2}{s}\right)\right] F^c\right.$$

$$+ \left[\frac{3}{2} c_2^2 + \frac{3}{2} c_3\, c_1 - 3\, \zeta_2\, c_3^2 + 3\, X_2 + \frac{\beta_0^2}{4} - \frac{3}{4} \beta_0 \left(c_2 - \frac{c_3}{2} \ln\left(\frac{\mu_R^2}{s}\right)\right)\right.$$

$$\left.\left. - \frac{3\beta_0}{4} c_3 \ln\left(\frac{M^2}{s}\right)\right] A^c + \frac{3}{2} c_3\, G^c + \frac{1}{2} K_3^c\right\} \mathcal{D}_2(x_{th})$$

$$+ \cdots$$

Here

$$X_3 = \frac{\beta_0}{12} c_3 - \sum_j C_j \frac{\beta_0}{24}$$

$$\begin{aligned} X_2 &= -\frac{\beta_0}{4} T_2 + \frac{\beta_0}{8} c_3 \ln\left(\frac{\mu_R^2}{s}\right) + \sum_i C_i \frac{K}{2} \\ &+ \sum_j C_j \left[-\frac{K}{4} + \frac{\beta_0}{8} \ln\left(\frac{M^2}{s}\right) \right] - \sum_j \frac{\beta_0}{8} B'^{(1)}_j \end{aligned}$$

$$\begin{aligned} X_1 &= \frac{\beta_0}{2} T_1 - \frac{\beta_0}{4} c_2 \ln\left(\frac{\mu_R^2}{s}\right) + \frac{\beta_0}{4} \zeta_2 c_3 + \sum_i \nu_i^{(2)} \\ &- \sum_i C_i \left[\frac{\beta_0}{8} \ln^2\left(\frac{\mu_F^2}{s}\right) - \frac{K}{2} \ln\left(\frac{\mu_F^2}{s}\right) - K \ln\left(\frac{-t_i}{M^2}\right) \right] \\ &+ \sum_j \left(B'^{(2)}_j + \nu_j^{(2)} \right) - \sum_j C_j \frac{\beta_0}{8} \zeta_2 + \sum_j \frac{\beta_0}{4} B'^{(1)}_j \ln\left(\frac{M^2}{s}\right) \\ &- \sum_j C_j \left[\frac{\beta_0}{8} \ln^2\left(\frac{M^2}{s}\right) - \frac{K}{2} \ln\left(\frac{M^2}{s}\right) \right] \end{aligned}$$

$$\begin{aligned} K_3^c &= \mathrm{tr}\left[H^{(0)} \left(\Gamma'^{(1)\,\dagger}_S\right)^3 S^{(0)} + H^{(0)} S^{(0)} \left(\Gamma'^{(1)}_S\right)^3 \right. \\ &\left. + 3\, H^{(0)} \left(\Gamma'^{(1)\,\dagger}_S\right)^2 S^{(0)} \Gamma'^{(1)}_S + 3\, H^{(0)} \Gamma'^{(1)\,\dagger}_S S^{(0)} \left(\Gamma'^{(1)}_S\right)^2 \right] \end{aligned}$$

Summary

- Soft and collinear logarithms at large x
- soft-gluon threshold corrections are important
- NNLO and NNNLO threshold corrections
- important for greater theoretical accuracy and reduction of scale dependence