High precision measurements and the minimal scale

Constraining models with Large eXtra Dimensions through high precision experiments

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Large eXtra Dimensions
The hierarchy problem: Gravity is much weaker than all other forces. Why?

\[ F_{\text{grav}} = G m_1 m_2 \frac{1}{r^2} = \frac{m_1}{m_{\text{Planck}}} \frac{m_1}{m_{\text{Planck}}} \frac{1}{r^2} \]

\[ \frac{m}{m_{\text{Planck}}} \ll 1 \text{ for all elementary particles.} \]
Motivation

Idea for a solution:

- Spacetime has more than 3+1 dimensions
- Gravity penetrates all dimensions, SM fields are bound to our submanifold
- Since we don’t see them, Extra dimensions are compactified, but with a larger radius
On distances below the compactification radius $R$, the higher-dimensional gravitational potential follows from Gauss’ law:

$$G(r) \sim \frac{1}{M_f^{d+2} r^d} \frac{m_2}{r^2} \quad (r \ll R)$$

For distances above, the gravitational force cannot further propagate into the extra dimensions, and the usual Newtonian potential is recovered:

$$G(r) \sim \frac{1}{m^2_{\text{Planck}}} \frac{m_2}{r^2} \quad (r \gg R)$$
The way it works

From the continuity condition at the compactification radius

\[ \frac{1}{M_f^{d+2} R^d} \frac{m_2}{R^2} = \frac{1}{m_{Planck}^2} \frac{m_2}{R^2} \]

follows the relation between the fundamental and the Planck scale:

\[ m_{Planck}^2 = M_f^{2+d} \]
The LXD model’s most remarkable feature is:

- The Planck scale is a derived, not a fundamental scale. The new fundamental scale $M_f$ is significantly smaller.
The Minimal Scale
FACT: Spacetime distance resolution has a lower limit!

This emerges from both

- String theory $T$-duality and
- combination of quantum theory and general relativity
Imagine a particle probed by another particle.

- To resolve a spacetime distance $a$, the probing particle must have wave number (and momentum) $p = k = 1/a$.
- As its energy increases, it disturbs the spacetime region it is supposed to probe.
- One obtains a lower limit, typically of order $l_{\text{Planck}} = 1/m_{\text{Planck}}$. 
Minimal Scale in LXDs

In models with Large eXtra Dimensions, the minimal scale is significantly higher!

\[ L_f \sim 1/M_f \]

This increased minimal length scale leads to observable phenomena.
Quantum theory with minimal scale
Full theory including both quantum theory and gravitation does not (yet) exist.

⇒ To achieve experimental prediction, the notion of a minimal length scale must be included manually into quantum theory.
Inclusion into quantum theory

How to do this?
Answer: Simply restrict allowed wavelengths to be greater than $L_f$, and thus:

$$k < \frac{1}{L_f}$$

Since momentum should not have an upper bound, this leads to a modification of the de Broglie-relations.
Inclusion into quantum theory

What do we expect from such a relation $k(p)$?

- upper limit $1/L_f$
- uneven (because of parity)
- linear behaviour for small $k$
Inclusion into quantum theory

A possible choice:

\[ L_f k^\mu(p^\mu) = \tanh \gamma \left( \frac{p^\mu}{M_f} \right)^\gamma \]
Inclusion into quantum theory

The new relation changes the basics of quantum theory:

Operators

\[
\begin{align*}
\langle x | \hat{x}^\mu | x \rangle &= x^\mu \\
\langle x | \hat{k}^\mu | x \rangle &= -i \frac{\partial}{\partial x^\mu} \\
\langle x | \hat{p}^\mu | x \rangle &= p \left( -i \frac{\partial}{\partial x^\mu} \right)
\end{align*}
\]
Inclusion into quantum theory

Commutators

\[
\langle x | [\hat{x}^\mu, \hat{k}^\nu] | x \rangle = i \delta^{\mu\nu}
\]

\[
\langle x | [\hat{x}^\mu, \hat{p}^\nu] | x \rangle = i \delta^{\mu\nu} \frac{\partial p^\mu}{\partial k^\mu}
\]

Normalisation

\[
\langle p' | p \rangle = \langle k(p') | k(p) \rangle = \delta (k(p') - k(p)) = \frac{\partial p}{\partial k} \delta (p' - p)
\]

Momentum space measure

\[
dp \rightarrow \frac{\partial k}{\partial p} dp
\]
Inclusion into quantum theory

Does it make sense?

Let’s look at some features one would expect from a theory with a minimal length:
Inclusion into quantum theory

Does it make sense?

Let’s look at some features one would expect from a theory with a minimal length:

The ’natural’ smooth cut-off

\[
\frac{\partial k}{\partial p} = \frac{1}{L_f} \frac{1}{\cosh^2\left(\frac{p}{M_f}\right)} = \frac{4}{L_f} \frac{1}{\left(\exp\left(\frac{p}{M_f}\right) + \exp\left(-\frac{p}{M_f}\right)\right)^2}
\]

\[
p \to \infty \quad \Rightarrow \quad \frac{4}{L_f} \frac{1}{\exp^2\left(\frac{p}{M_f}\right)}
\]
**Inclusion into quantum theory**

**The uncertainty relation**

\[ \Delta p \Delta x \geq \frac{1}{2} \left| \frac{\partial p}{\partial k} \right| = \frac{1}{2} \frac{1}{1 - L_f^2 k^2(p)} \]

![Graph](image)
High precision measurements
First-order approximation

Since we are interested in the first effects, we approximate the $p(k)$ relation for small momentum:

$$p^\mu (k^\mu) \approx \left(1 + \frac{1}{3} L_f^2 |k^\mu|^2 \right) k^\mu$$

The resulting first-order momentum operator (in one dimension) is:

$$\langle x|\hat{p}|x\rangle \approx -i \left(1 - \frac{1}{3} L_f^2 \frac{\partial^2}{\partial x^2} \right) \frac{\partial}{\partial x}$$
The Hydrogen S1-S2 transition energy is known to very high precision. The correction of the energy levels due to the minimal length scale is given by:

\[ E_n \approx -\frac{e^4 m}{2n^2} \left( 1 - \frac{e^4 m^2}{M^2_f n^2} \right) \]
The Hydrogen atom

A plot of the relative deviation:
The muon gyromagnetic moment

The inclusion of the minimal length modifications into the Dirac equation

\[ i \frac{\partial}{\partial t} \psi(\vec{r}) \approx (-i \left(1 - \frac{1}{3} L_f^2 m^2 \right) \vec{\alpha} \cdot \vec{\nabla} + \left(1 + \frac{1}{3} L_f^2 (\Delta - m^2) \right) \beta m) \psi(\vec{r}) \]

leads (after some algebra) to a modified gyromagnetic moment:

\[ g \approx g_{SM} \left(1 - \frac{1}{3} \frac{m^2}{M_f^2} \right) \]
The muon gyromagnetic moment

The Standard Model and experimental data:

\begin{center}
\begin{tikzpicture}
\begin{axis}[
    width=\textwidth,
    height=0.5\textwidth,
    xlabel={\text{SM, e$^+$e$^-$, SM, $\tau$, Exp., BNL, Exp., $\varnothing$}},
    ylabel={$(a_{\mu} - 11659000) \cdot 10^{10}$},
    ymin=165, ymax=225
]
\addplot[only marks, mark size=2pt] coordinates {
    (1, 190), (2, 180), (3, 175), (4, 165),
    (5, 205), (6, 210), (7, 215), (8, 220),
    (9, 225)
};
\end{axis}
\end{tikzpicture}
\end{center}
The present data clearly indicates that modifications due to effects beyond the Standard Model (which includes the minimal length scale) have to be smaller than \(10^{-8}\). This leads to a limit on the fundamental scale:

\[
M_f \gtrsim 500 \text{ GeV}
\]
Conclusions

• In models with LXDs, the minimal scale increases significantly and thus has to be considered when making experimental predictions.

• It provides a new mean of constraining those models.

• Present data on the muon gyromagnetic moment already sets a limit on the fundamental scale.